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Marx-Okishio System and Perron-Frobenius Theorem

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Abstract. This paper presents two notes on Marx's political economy. First, in the case of the so-called Leontief framework, one sees that major propositions of Marx's political economy are all derived from the Perron and Perron-Frobenius theorems. Since the results owe much to Okishio(1963, 1993), the system will be named the Marx-Okishio system. Second, one will see that thePerron theorem is derived from the convergence of the transformation procedure. The main point of this paper is that Marx-Okishio's system is fundamentally equivalent to the Perron theorem. Remark that, by applying Morishima's dual-dualities, proofs will be made simpler.

***Keywords*:** Marx-Okishio, Perron theorem, Perron-Frobenius theorem, Morishima, dual dualities, matrix power, convergence.

1. **Introduction**

Marx's main literature, *Das Kapital*, contains many important quantitative problems, such as the determination of values of commodities, the analysis of surplus values and exploitation, the transformation problem, and problems related to the profit rate *etc*. Many a scholar challenged those problems, and it is not too much to say that Okishio(1963) was the first in presenting the systematic analysis and mathematical description of Marx's political economy. This development is called *Marx-Okishio's system* hereafter.

Okishio's formulation is based on the system without fixed capital, joint production and skilled labour. He started his mathematical discussion from the Perron theorem(1907) and the Perron-Frobenius theorem(1912), hereafter P-theorem and PF-theorem respectively, concerning the existence of the positive eigenvalue accompanied by positive eigenvectors of positive or non-negative irreducible matrices. Basic propositions of Marx, however, were almost all supported by Okishio. In this sense, Okishio established the Marx-Okishio system.

Marx-Okishio's theory was later supplemented by Morishima (1973) and properly characterised as a system of *dual dualities*. In what follows the idea of dual dualities is explicitly employed in proving propositions.

Section 3 is dedicated to demonstrating that the convergence of Marx-Okishio's transformation implies the PF-theorem, with an elementary approach.

In Section 4, the summary of this paper will be stated.

Appendix A gives some preliminaries on non-negative matrices.

2. Marx-Okishio's System from the Angle of Dual Dualities

2.1. Mathematical preliminaries. Assume the Leontief input-output framework of production with n types of commodities. Let $A \geq O$ and $\ell > 0_n$ denote an irreducible input matrix and a labour vector, respectively. The most fundamental assumption with respect to the system is the productiveness of the system matrix A, that is,

A.1. There exists an $x > 0^n$ such that $x > Ax$.

This is equivalent to $\lambda_A < 1$ with respect to a non-negative and irreducible A.

2.2. Dual Dualities. The term *dual dualities* was first employed by Morishima (1973, p.106), and it elegantly represents the system that covers Marx-Okishio's. Three systems, that is, the value system, the production price system, and the capacity growth system will be summarised here.

The value system. The system of *values*, w, of commodities is described by

$$w = wA + \ell. \tag{2.1}$$

PROPOSITION 2.1. (1) A.1. *implies that w satisfying* (2.1) *is positive.*
(2) *For x such that $x = Ax + y$ with $y > 0$, $wy = \ell x$.*

Surplus value and surplus labour. Let $f \geq 0^n$ stand for a *wage-goods bundle* for a unit expenditure of labour. The *value of labour-power* is then determined by wf. $1 - wf$ gives the magnitude of *surplus value* per unit of labour. The *rate of surplus value* is then defined by the following:

$$\mu = \frac{1}{wf} - 1. \tag{2.2}$$

Let z denote the vector of *necessary products* that is necessary for reproducing labour power expended for the production of current products $x > 0^n$, and

$$z = Az + f\ell x. \tag{2.3}$$

The amount of *necessary labour* is now defined by ℓz, and the rate of *surplus labour* is defined by $\eta = \frac{\ell x}{\ell z} - 1$. From a simple manipulation, one has:

PROPOSITION 2.2. $\eta = \mu$.

Profitability. Since the value of labour power enters into capital cost, the magnitude of surplus value appears as *profit*, and values of commodities emerge as *prices* of commodities to the eyes of capitalists. What matters to capitalists is whether the system of commodity production is profitable: if there exists a $p > 0_n$ such that

$$p > pA + pf\ell = p(A + f\ell), \tag{2.4}$$

the system (A, ℓ, f) is called *profitable*. Write the *extended input matrix* $M = A + f\ell$, and the system of commodity production is profitable *iff* $\lambda_M < 1$.

Remark that from (2.1) and (2.2) one has here:

PROPOSITION 2.3. $\mu > 0$ *implies that M is profitable.*

The transformation of values into production prices. In Marx's political economy, the value system is transformed to the production price system that represents the formation of the *average*, or the *equilibrium rate of profit* on advanced capital.

In what follows, the next one is assumed:

A.2. There exists an $x > 0^n$ such that $x > Mx$.

Marx-Okishio's iteration procedure is described as follows.

For an arbitrarily given $x > 0^n$, one can generate sequences, $\{\pi_k\}$ and $\{w^k\}$, such that, with the initial value $w^o = w$,

$$1 + \pi_k = \frac{w^k x}{w^k M x}, \tag{2.5}$$

$$w^{k+1} = (1 + \pi_k) w^k M. \tag{2.6}$$

THEOREM 2.4. *The sequences generated by* (2.5) *and* (2.6) *converge, if M is primitive with* A.2.

In fact, one may take a right-side PF-vector x such that $Mx = \lambda_M x$ in view of Theorem A.1. Then, $(1 + \pi_k)\lambda_M = 1$, namely, $\{\pi_k\}$ is convergent: $\lim_{k\to\infty} \pi_k = \pi$. Now, one has only to look at

$$w^{k+1} = (1 + \pi) w^k M = w^o \tilde{M}^k, \tag{2.7}$$

where $\tilde{M} = (1 + \pi)M$ with $\pi = \frac{1}{\lambda_M} - 1$ and an initial value $w^o = w$.

As explained in Appendix A, the matrix power $\tilde{M}^k$, and hence $\{w^k\}$ converges.

The limits of the sequence $\{\pi_k\}$ and that of $\{w^k\}$ define the system of *production prices* of commodities.

The system of production prices. In summary, the system of production prices of commodities is represented statically by the following: a pair of (p, π) such that

$$p = (1 + \pi) pM, \tag{2.8}$$

where $\pi = \dfrac{1}{\lambda_M} - 1$, and $px = wx$ for an arbitrary chosen $x > 0^n$.

Remark that p is normalised on the basis of the value system w.

The gist of Marx's political economy is given by the following:

THEOREM 2.5 (Fundamental Marxian Theorem). $\pi > 0$ *iff* $\mu > 0$.

Similarly, one obtains from (2.3) and (2.8):

PROPOSITION 2.6. $\pi > 0$ *iff* $\eta > 0$.

The capacity growth path. The dual system of the production price system in the quantity side is represented by the following:

$$x^c = (1 + \delta_c) M x^c, \tag{2.9}$$

where δ_c and x^c stand for, respectively, the *capacity growth rate* and the *capacity growth path*, which appeared in the above paragraph substantively.

PROPOSITION 2.7. *In* (2.8) *and* (2.9), $\delta_c = \pi$.

2.3. **The Okishio theorem.** The Okishio theorem states that cost-reducing innovation will bring about a higher profit rate when the innovation prevails in the economy. In what follows, process innovation alone will be dealt with.[1]

Let M and $\bar{M}$ stand for old and new systems of techniques, respectively. Let p denote a production price vector corresponding to M. Suppose that $\bar{M}$ is cost-reducing under p, and one has

$$pM \geq p\bar{M}. \tag{2.10}$$

PROPOSITION 2.8 (generalised Okishio's theorem). *Let M and $\bar{M}$ be irreducible, non-negative matrices. Suppose that, for p such that $pM = \lambda_M p$,* (2.10) *holds true. Then, $\lambda_{\bar{M}} < \lambda_M$.*

In fact, take $u = x^c_{\bar{M}} > 0^n$, and one has

$$pMu = \lambda_M pu > p\bar{M}u = \lambda_{\bar{M}} pu. \tag{2.11}$$

Since $pu > 0$, it follows that $\lambda_{\bar{M}} < \lambda_M$.

The original Okishio theorem assumes that the real wage rate is kept constant throughout the innovation process. Original Okishio theorem corresponds to one possible case in which $M = A + f\ell$ and $\bar{M} = \bar{A} + f\bar{\ell}$, where $\bar{X}$ denotes an innovative technique matrix or vector. Needless to say, f might be changed, say $\bar{f}$, in so far as $pf = p\bar{f}$. All possible cases are included in Proposition 2.8.

The Okishio theorem says that the production prices function as an important capitalists' tool for choosing techniques.

3. From Marx-Okishio to Perron-Frobenius

Take the sequences generated by (2.5) and (2.6), and suppose that they converge.

THEOREM 3.1. *If the Marx-Okishio transformation procedure with an irreducible system matrix converges, the system matrix is primitive, and the Perron theorem follows.*

In fact, the sequence $\{\pi_k\}$ possesses its limit, say π, so that one has only to consider the sequence $\{w^k\}$. Since the property of M and its transpose is the same, one may consider an iteration

$$x^t = \tilde{M}x^{t-1}, \tag{3.1}$$

where $\tilde{M} = (1+\pi)M$ with $x^o > 0^n$. Note that for given $x > 0^n$, one has for any $k > 0$

$$w^k x = w^{k+1} x. \tag{3.2}$$

[1] As for product-innovation, refer to, *e.g.*, Fujimori(1998).

Since it converges, it follows that M possesses the dominant eigenvalue and a positive eigenvector associated with it, which conforms to the case of matrix-power convergence *à la* Jordan's cannonical form.

This means that non-negative irreducible matrices possess a positive eigenvalue accompanied by a positive eigenvector. The core part of the PF-theorem is proved.

4. **Concluding Remarks**

So far, two major points are summarised and demonstrated. The *first* one is that most of propositions of Marx-Okishio's system are derived from the PF-theorem. Primitivity of nonnegative coefficient matrices should be assumed, however, for the convergence of the transformation procedure. The *second* point is the other way round. If one starts from the convergence of Marx-Okishio's transformation of values into production prices, one can prove the PF-theorem. That is to say, *Marx-Okishio's system is equivalent to the Perron theorem*. One may say that the transformation is the apex in Marx's political economy.

One should remark that primitivity of matrices is not a strong condition in dealing with linear models of closed economies. Such an economy includes the rice growing sector, for instance, with recursive inputs of its own outputs.

The proof itself can be done without employing the capacity growth path, as was done in original Okishio's paper (Okishio, 1963, 1993), except Proposition 3.1. Some may say that Morishima's dual-dualities is redundant or not essential with respect to the framework of theory itself. If one introduces Morishima's idea of dual dualities, however, one can make use of the quantity system, in particular the capacity growth path, as an aggregator, and the proofs of propositions can be made simpler and straight. The recognition of dual dualities is an important contribution made by Morishima(1973) in this sense. This is why this paper explicitly applied Morishima's dual-dualities.

One should recall that Uzawa (1962) proved the equivalence between Walras' theorem of existence of general equilibrium and Brouwer's fixed point theorem. Uzawa's equivalence theorem shows that Walras' theory is robust as a vehicle of rigorous and logical arguments. Likewise, it is seen that Marx-Okishio's system is logically robust.

Okishio's propositions, in particular Okishio's theorem, were challenged by many scholars, but most of those critiques are misleading, because they do not understand well the mathematical background of Marx-Okishio's theory.

Appendix A

A.1. **Preliminaries.** In most of the cases, square matrices are dealt with. As for vector- and matrix-notations, 0^n and 0_n denote, respectively, the n-dim zero column- and row- vectors, and O stands for the zero matrix of an appropriate dimension. In matrix- and vector-inequalities, the three elementwise inequalities, $>$ (strict inequality), $\geq$ (semi-inequality, $\neq$) and $\geqq$ are distinguished. The same applies to the opposite inequalities. As for terminology and details, refer to Nikaido(1968), Seneta(1980), Strang(1988), Meyer (2004), Chatelin(2012) *etc.*

A.2. **Theorems on non-negative matrices.** The major mathematical propositions that support the following discussion are two: one is P-theorem on positive matrices, and the other PF-theorem on non-negative irreducible matrices. The core part of the theorems can be stated as follows:

THEOREM A.1 (Perron-Frobenius). *Let $A \geq O$ be irreducible.* (1) *There exists a positive eigenpair $(\lambda_A, x) > (0, 0^n)$, such that $\lambda_A x = Ax$.*
(2) *λ_A is a simple root of the characteristic equation of A.*
(3.a) *For other eigenvalue $\lambda_i \neq \lambda_A$ holds $\lambda_A \geq |\lambda_i|$.*
(3.b) *If $A > O$, then a strict inequality holds $\lambda_A > |\lambda_i|$.*

The items (1), (2) and (3.a) (*resp.* (3.b)) constitute the core of F- (*resp.* P-)theorem.

P- theorem can be easily extended to $A \geq O$ such that for some $m > 0$, $A^m > O$.

Remark the difference between the two, (3.a) and (3.b). F-theorem permits the existence of eigenvalues on the same spectral radius. This is an important point.

In this paper, λ_A denotes the *Perron-Frobenius root*, in short, *PF-root*, of A.

One has to deal with the limit of matrix-power A^k.

Given a general $m \times n$ matrix sequence $\left\{A^{(k)}\right\}, k = 1, 2, \cdots$, if for any i, j, sequence $\left\{a_{ij}^{(k)}\right\}$ is convergent, the matrix sequence $\left\{A^{(k)}\right\}$ is said to be *convergent*.

For a special matrix sequence $\left\{A_{n\times n}^k\right\}, k = 1, 2, \cdots$, however, there is another way to judge its convergence: one needs to employ the Jordan canonical form.

A.3. **Jordan's canonical form.**

PROPOSITION A.2 (Jordan's canonical form). *Let A denote an $n \times n$ matrix. There exists an $n \times n$ non-singular matrix P, such that*

$$P^{-1}AP = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix},$$

where $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{d_i \times d_i}$ *is a Jordan block, λ_i is the eigenvalue of A, and d_i is the multiplicity of λ_i for A.*

Then, we have

$$A^k = P \operatorname{diag}\left(J_1^k, \cdots, J_m^k\right) P^{-1}.$$

PROPOSITION A.3. *The matrix A^k is convergent, iff $J_1^k, \cdots, J_m^k$ converge.*

We know by matrix multiplication that

$$J_i^k = \begin{pmatrix} \lambda_i^k & C_k^1 \lambda_i^{k-1} & \cdots & C_k^{d_i-1} \lambda_i^{k-d_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & C_k^1 \lambda_i^{k-1} \\ & & & \lambda_i^k \end{pmatrix}_{d_i \times d_i},$$

where

$$C_k^l = \begin{cases} \dfrac{k(k-1)\cdots(k-l+1)}{l!}, & \text{if } l \leqslant k, \\ 0, & \text{if } l > k. \end{cases}$$

If $|\lambda_i| < 1$, then J_i^k will converge to the zero matrix.

If $|\lambda_i| > 1$, then J_i^k will diverge.

If $|\lambda_i| = 1$, we need further discussion. Since the elements of the upper triangle are divergent, if J_i^k is convergent, the multiplicity of λ_i must be 1, that is, the matrix J_i degenerates into the first-order matrix $[\lambda_i]$. In addition, λ_i must be 1, because negative numbers and imaginary numbers cannot converge.

Now, we can summarize the convergence conditions for matrix sequence $\{A_{n\times n}^k\}$.

PROPOSITION A.4. *A matrix sequence* $\{A_{n\times n}^k\}$ *is convergent, if either of the following two holds true*:

(1) *The spectral radius of A is less than* 1.

(2) *The unity,* 1*, is an eigenvalue of A,*[2] *and the absolute value of other eigenvalues is strictly less than* 1.

A.4. **Convergence of the power method.** Based on the Jordan canonical form of matrices, one can obtain the following:

PROPOSITION A.5. *Let A stand for an n-dim square matrix, with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *such that* $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$. *A sequence* $\{x^t\}$, *generated by*

$$x^{t+1} = Ax^t, t = 0, 1, \ldots \tag{A.1}$$

converges to $c\lambda_1^t \xi$, *where* $A\xi = \lambda_1 \xi$ *and c is an appropriately chosen scalar.*

The power method fails, however, when the coefficient matrix is not primitive.

A.5. **A tip for the imprimitive case.** If an irreducible matrix A is imprimitive, then A^k does not converge, so that some tricks should be introduced to prove F-theorem from P-theorem.

First, confirm that one has the following:

PROPOSITION A.6. *Let* $A = \{a_{ij}\} \geq O$ *stand for an n-dim irreducible matrix. If, for some* $i = 1, .., n$, $a_{ii} > 0$, *then A is primitive.*

[2] The algebraic multiplicity of 1 is equal to the geometric multiplicity.

Now, take an irreducible but imprimitive A. Consider a sequence of $\{A_k\}$, such that $A_k = \frac{1}{k}I + A$. A_k is primitive. Take the sequence of eigenpairs $\{(\lambda_k, x^k)\}$, $k = 1, 2, \ldots$, where $x^k > 0^n$ is normalised as $||x^k|| = 1$. Since $\{A_k\}$ is decreasing, $\{\lambda_k\}$ is decreasing and bounded from below, $\lambda_k \geq 0$. As $k \to \infty$, $x^k \to x > 0^n$, and hence $\lambda_k \to \lambda_A > 0$. Thus, F-theorem can be derived from P-theorem.

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