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Analyses of Equilibrium Shifts and Stability Conditions on the Phillips-Type Policy in Business Cycle Models with Capital Accumulation

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ABSTRACT. This paper extends Phillips' stabilization policy framework to the case of economies with capital accumulation. We consider two investment functions based on Kaleckian and Keynesian ideas. Time-lags of capital production and policy are taken into account. Applying the Laplace transform, we develop a framework of stability analysis. We investigate shifts in equilibrium level caused by an exogenous demand shock and policy implementation. Furthermore, we derive necessary and sufficient conditions for asymptotic stability from the Routh-Hurwitz theorem. It is found that the existence of capital accumulation does not affect the stationary level and the Phillips-type policy is still effective. We discuss economic implications of each stability condition for investment functions of Kalecki-type and Duesenberry-type, briefly revisiting a discussion on the stability of Kalecki's early business cycle model given by Lange (1970).

JEL Classification: C62, E11, E12, E61

Keywords: Phillips-type stabilization policy; Kalecki's business cycle model; Laplace transform

1. Introduction

The main purposes of this article are to evaluate stabilization effects of the Phillips-type policy under capital accumulation and to reveal stability conditions. The theory of stabilization policy for a multiplier-accelerator model proposed by Phillips (1954) is acknowledged as a pioneering work on economic stabilization theory. Influences from Phillips' theory can be seen in studies on feedback-rule policies, such as Taylor's policy rule (Taylor (1968, 1993)). Although Samuelson (1939) had previously argued that the combination of the multiplier effect and the accelerator principle generates explosive business cycles, Samuelson's model cannot indicate stabilizing effects of the rule-based policy, because it treats government spending as

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an exogenous variable. On the other hand, Phillips' multiplier-accelerator model endogenizes government spending as a function that depends on the production level. Phillips focused on a restoring process of stationary equilibrium from a temporary decline in production levels caused by an exogenous demand shock. He demonstrated that, even in the case where there exists a time-lag in the effect of policies, the initial equilibrium can be stably restored through a feedback-rule policy which consists of proportional, integral, and derivative terms.

As stabilization policy theories develop, Phillips' original framework has been replaced by the methods of optimal linear-quadratic stabilization theory.[1] Nevertheless, the Phillips-type policy, whose independent variable is only the production level, has the advantage that it is more simple and realistic than optimal policies assuming perfect information on structural parameters of an economy. Lucas (1976) was concerned that any change in a policy itself will change the structure of the economy, but Philips' policy could still be effective as a stabilization policy even if the economic structure changed in that way. This is because changes in policy rule, equivalent to changing three policy parameters of policy functions within Philips' framework, do not necessarily impair the stabilizing effect of the policy. In order to guarantee the stabilization under circumstances where such changes in the economic structure can occur, it is necessary to clarify the relation between policy parameters and economic structural parameters to be satisfied for stability. However, Phillips did not show the allowed range of policy parameters to ensure the stability of equilibrium. Moreover, since Phillips' multiplier-accelerator model treats only flow variables, we must clarify whether the Phillips-type policy can stabilize an economy in which capital accumulation explicitly affect its stability.

To construct a macrodynamic model with capital accumulation, we firstly focus on the business cycle model of Kalecki (1935), which predates Keynes's *General Theory*. Kalecki's model can explicitly treat the influence of capital accumulation on economic stability and is still attractive for research on cyclical behaviors of economy;[2] nevertheless little attention has been paid to the stabilization problem within Kalecki's setting. Lange (1970) focused on a necessary condition for asymptotic stability in Kalecki's model in order to discuss the stability of a capitalist economy. This stability condition says that, in order for the economy to be stable, it is necessary for the magnitude of investment decisions to be less than the total accumulation. Lange emphasized as follows:

[1] See, *e.g.*, Sengupta (1970), Turnovsky (1973), and Turnovsky (2011).

[2] See, *e.g.*, Krawiec and Szydlowski (1999), Szydłowski (2002), Szydłowski and Krawiec (2005), Ballestra, Guerrini, and Pacelli (2013), and Ercolani (2014).

> The increasing trend, i.e. the expanded reproduction of investment goods, is paid for by the instability of the system. The stability of the system, on the other hand, is paid for by simple or less than simple reproduction. This is, according to Kalecki's model, the basic dilemma of the capitalist system: growth and instability, or stability and stagnation or even decline. (Lange, 1970, pp.140-1).

However, he goes no further than implying a possibility that some compensating factors can exogenously support the stable full reproduction. Thus, the following questions arise: Can the Phillips-type policy be such a compensation factor? Will Lange's dilemma be resolved through implementation of policy?

In this paper, we tackle the above questions by analyzing equilibrium shifts and stability conditions on macrodynamic models with the Phillips-type policy. Section 2 provides an extension of Phillips' framework to include a capital stock, decisions of investment, and a time-lag associated with producing capital goods. It is also of importance to investigate influences of the difference between ideas of Kaleckian and Keynesian on investment decisions on economic stability. We also take a Duesenberry-type investment function into account as a case of Keynesian view in Section 3. Within these settings, we focus on a restoring process of stationary equilibrium from a temporary decline in production level. Further reformulations by taking the Laplace transform are presented such that initial value problems of differential equations are rigorously solved. The Laplace transform method is known as a rigorous justification of Heaviside's operational calculus by Bromwich (1917), Carson (1926), and other mathematicians. Since Phillips merely defined time-lags in terms of Heaviside's differential operator, our formulae provide a rigorization of his model. Section 4 demonstrates the performance of the Phillips-type policy to offset shifts in equilibrium levels for each of the investment functions. In Section 5, we derive the stability conditions by applying the Routh-Hurwitz theorem.[3] We discuss the effects of structural parameters and policy parameters on stability, comparing the results of the Kalecki-type investment function with those on the Duesenberry-type.

2. A Model with a Kalecki-Type Investment Function

2.1. Description in the time domain. In this section, we provide a macrodynamic model of an economy with capital accumulation and policy implementation. This section treats an investment function in accordance with Kalecki (1935), but our

[3] Mathematical results on Proposition 4.1, 4.3, 5.1 on a Kalecki-type investment function are already obtained in Kageyama (2016). The present study provides a more detailed characterization of the stability conditions and novel mathematical results on a Duesenberry-type investment function.

description follows an interpretation of Kalecki's model by Allen (1966) and Lange (1970). We also introduce the Phillips-type policy and an exogenous demand shock into the model.

The economic variables are functions of continuously varying time t. The amount of production at time t, $Y(t)$, is the sum of consumption $C(t)$, net investment $I(t)$, autonomous demand $A(t)$, and government spending $G(t)$:

$$Y(t) = C(t) + I(t) + A(t) + G(t). \tag{2.1}$$

For simplicity, it is assumed that the market clearing condition is satisfied. As in Phillips (1954), the initial production level is taken as a reference, $Y(0^-) = 0$, and hence $Y(t)$ denotes the deviation between the initial and the current level of production. The autonomous demand is an exogenous variable as a disturbing factor for the initial equilibrium state.

The amount of consumption is proportional to production:

$$C(t) = (1-l)Y(t), \tag{2.2}$$

where l is a positive constant representing the saving rate, $0 < l < 1$. $lY(t)$ is equivalent to the amount of savings.

Let us introduce capital stock, investment decision, and capital production into the model. $L(t)$ represents the amount of net production of capital goods at time t. $K(t)$ represents the amount of stock of capital goods. The relation between $K(t)$ and $L(t)$ is expressed by

$$K'(t) = L(t), \tag{2.3}$$

where the prime symbol denotes the derivative with respect to t. We have ignored depreciation of capital. The initial conditions are given by $K(0^-) = L(0^-) = 0$.

$B(t)$ represents the amount of planned investment at time t. $B(t)$ is determined by the following equation:

$$B(t) = alY(t) - bK(t), \tag{2.4}$$

where a and b are positive constants. This relation says that the decision of investment is influenced directly by savings and inversely by the existing stock of capital goods. The coefficients a and b express the sensitivities of investment decision to savings and to the stock of capital goods, respectively. Kalecki originally assumes that only capitalists make savings and decide the amount of investment, but in Lange's interpretation, non-capitalists' savings influence the decisions to invest. For simplicity, we assume that the amount of net investment is equal to that of investment decision,[4] *i.e.*,

$$B(t) = I(t) = alY(t) - bK(t), \tag{2.5}$$

with the initial conditions $B(0^-) = I(0^-) = 0$.

A time-lag in the production of capital goods is given by a continuously distributed lag as follows:

$$L'(t) = \frac{1}{\tau_L}(B(t) - L(t)), \tag{2.6}$$

where τ_L is a positive constant representing the velocity of the lag of capital goods production.[5] The expression $B(t)-L(t)$ corresponds to the amount of capital goods in course of construction. Increasing the values of τ_L means that a time-lag of capital goods production becomes longer.

We introduce the policy rule proposed by Phillips (1954) into the model with the object of offsetting the deviation from the initial stationary level and of suppressing oscillations in production level generated by an exogenous change in demand. Phillips proposed the following policy function as a rule of government spending:

$$P(t) = -\mu_1\left(Y(t) - Y^d\right) - \mu_2\int_0^t \left(Y(\tau) - Y^d\right)d\tau - \mu_3\left(Y'(t) - Y^d\right),$$

where μ_1, μ_2 and μ_3 are positive constants and Y^d is any desired production level. As in Phillips (1954), the initial stationary equilibrium level is regarded as a desired stationary level, namely $Y^d = Y(0^-) = 0$. Thus, we use

$$P(t) = -\mu_1 Y(t) - \mu_2\int_0^t Y(\tau)d\tau - \mu_3 Y'(t). \tag{2.7}$$

[4] In Kalecki (1935), the payments of investment are spread throughout the period of production of capital goods. The investment at time t is equal to the average value of all investment ordered during the time period $[t-\theta, t]$, *i.e.*,

$$I(t) = \frac{1}{\theta}\int_{t-\theta}^t B(\tau)d\tau.$$

The amount of investment ordered during this period is $\int_{t-\theta}^t B(\tau)d\tau$, and $\frac{1}{\theta}$ of the total of orders is completed. We ignore such a spread of payment.

[5] In Kalecki (1935), it is assumed that the production of capital goods takes a time period θ to complete the net production of capital goods. The net production of capital goods at time t equals the investment planned at time $t-\theta$, *i.e.*,

$$L(t) = B(t-\theta).$$

As a substitute for this discrete lag, we use the continuously distributed one. In (2.6), if $B(t)$ is a constant B, we obtain a solution of (2.6) given by an exponential function, *i.e.*, $L(t) = B\left(1 - e^{-t/\tau_L}\right)$. As $t \to \infty$, we obtain $L(t) = B(t)$.

Thus, the target of the stabilization policy is to achieve asymptotic stability, namely, $\lim_{t\to\infty} Y(t) = 0$.

A policy lag until demand is affected is supposed as

$$G'(t) = \frac{1}{\tau_G}(P(t) - G(t)), \tag{2.8}$$

where τ_G is a positive constant representing the velocity of policy lags. Increasing the value of τ_G means that a time-lag of policy becomes longer.

An exogenous change in autonomous demand to a constant level that happens at time 0 is written as

$$A(t) = Au_0(t), \tag{2.9}$$

where

$$u_0(t) = \begin{cases} 1, & t > 0, \\ \frac{1}{2}, & t = 0, \\ 0, & t < 0, \end{cases}$$

and A is a constant.

Note that since the value of each economic variable is considered as the amount of deviation from the initial equilibrium level, it does not represent the total amount. As in Phillips (1954), we define that the economic variables can take negative values. For the above reason, this definition is not necessarily unrealistic. If needed, we can restrict the target level Y^d and autonomous demand $A(t)$ to be positive.

2.2. The Laplace transform of the model. This section restates the model developed in Section 2.1 in terms of the Laplace transform. The Laplace transform converts functions in the time domain into those in the complex domain, giving a simple but rigorous treatment of the model composed of differential equations (see Appendix for details).

Let s be a complex variable and let $f(t)$ be a real-valued function of a real variable t. Then, the Laplace transform of $f(t)$ is defined by

$$\hat{f}(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt = \lim_{\substack{\varepsilon\to 0 \\ b\to\infty}} \int_{-\varepsilon}^{b} f(t)e^{-st}dt \tag{2.10}$$

where ε is positive. It is easy to see that if $f(t)$ is piecewise continuous on every finite interval in $[0, \infty)$ and of exponential order,[6] then the Laplace transform exists.

[6] $f(t)$ is said to be of exponential order if there exist real constants M and γ such that

$$|f(t)| \le Me^{\gamma t}.$$

We assume that all the economic variables are of exponential order and continuous. With the linearity of the Laplace transform such that $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1\mathcal{L}[f_1(t)] + a_2\mathcal{L}[f_2(t)]$, we can describe the Laplace transforms of economic variables as follows:[7]

$$\hat{Y}(s) = \hat{C}(s) + \hat{I}(s) + \hat{A}(s) + \hat{G}(s), \tag{2.11}$$

$$\hat{C}(s) = (1-l)\hat{Y}(s), \tag{2.12}$$

$$s\hat{K}(s) = \hat{L}(s), \tag{2.13}$$

$$\hat{B}(s) = al\hat{Y}(s) - b\hat{K}(s), \tag{2.14}$$

$$\hat{B}(s) = \hat{I}(s), \tag{2.15}$$

$$\hat{L}(s) = \frac{1}{\tau_L s + 1}\hat{B}(s), \tag{2.16}$$

$$\hat{G}(s) = \frac{1}{\tau_G s + 1}\left(-\mu_1\hat{Y}(s) - \mu_2\frac{1}{s}\hat{Y}(s) - \mu_3 s\hat{Y}(s)\right). \tag{2.17}$$

From equations (2.13) - (2.16), we obtain

$$\hat{I}(s) = \frac{als(\tau_L s + 1)}{s(\tau_L s + 1) + b}\hat{Y}(s). \tag{2.18}$$

Equations (2.11), (2.12), (2.17), and (2.18) are summarized as follows:

$$\hat{Y}(s) = \frac{N(s)}{D(s)}\hat{A}(s), \tag{2.19}$$

where

$$N(s) = \tau_L\tau_G s^4 + (\tau_L + \tau_G)s^3 + (1 + b\tau_G)s^2 + bs, \tag{2.20}$$

$$\begin{aligned} D(s) = {} & [\tau_L\tau_G l(1-a) + \mu_3\tau_L]\, s^4 \\ & + [(\tau_L + \tau_G)l(1-a) + \mu_1\tau_L + \mu_3]\, s^3 \\ & + [l(b\tau_G + 1 - a) + \mu_1 + \mu_2\tau_L + \mu_3 b]\, s^2 \\ & + [b(l + \mu_1) + \mu_2]\, s + \mu_2 b. \end{aligned} \tag{2.21}$$

We eventually obtain the Laplace transform of $A(t) = Au_0(t)$ as follows:

$$\hat{A}(s) = \mathcal{L}[Au_0(t)] = A\int_{0^-}^{\infty} u_0(t)e^{-st}dt = A\int_{0^-}^{\infty} e^{-st}dt = \frac{A}{s}. \tag{2.22}$$

[7] Equation (2.16) is obtained from $s\hat{L}(s) = \frac{1}{\tau_L}\left(\hat{B}(s) - \hat{L}(s)\right)$. Equation (2.13) is obtained from $\mathcal{L}[K'(t)] = s\mathcal{L}[K(t)]$. Equation (2.17) is obtained by using the following relation:

$$\mathcal{L}[Y(t)] = \mathcal{L}\left[Q'(t)\right] = s\mathcal{L}[Q(t)] - Q(0^-) = s\mathcal{L}\left[\int_0^t Y(\tau)d\tau\right],$$

where $Q(t) = \int_0^t Y(\tau)d\tau$.

3. A Model with a Duesenberry-Type Investment Function

3.1. Description in the time domain. In Kalecki's line, the amount of investment decisions described by (2.4) depends on the amount of savings. This relation comes from the assumption that only capitalists make savings and decide the amount of investment in Kalecki (1935). Allen (1966) and Lange (1970) read the term of capitalists' savings as non-capitalists' ones. Such a description of investment decisions is indeed valid in some cases. However, in Keynes's principle of effective demand, it is supposed that investment is unconstrained by savings. With this understanding of the meaning of the principle of effective demand, we may write investment decisions as

$$B(t) = aY(t) - bK(t). \tag{3.1}$$

This type of investment function is found in a business cycle model proposed by Duesenberry (1958), which is known as a generalized multiplier-accelerator model in discrete-time.[8] For simplicity, we assume that $B(t) = I(t)$. Thus, we use

$$I(t) = aY(t) - bK(t), \tag{3.2}$$

instead of (2.5). Then, we can construct a model with (3.2), leaving the equations (2.1), (2.2), (2.3), (2.6), (2.7), (2.8), (2.9) unchanged.

3.2. The Laplace transform of the model. From (2.13), (2.16) and (3.2), we obtain

$$\hat{I}(s) = \frac{as(\tau_L s + 1)}{s(\tau_L s + 1) + b}\hat{Y}(s). \tag{3.3}$$

Then, $\hat{Y}(s)$ can be written as

$$\hat{Y}(s) = \frac{N(s)}{D(s)}\hat{A}(s), \tag{3.4}$$

where

$$N(s) = \tau_L \tau_G s^4 + (\tau_L + \tau_G)s^3 + (1 + b\tau_G)s^2 + bs, \tag{3.5}$$

$$\begin{aligned} D(s) = {} & [\tau_L \tau_G (l - a) + \mu_3 \tau_L]\, s^4 + [(\tau_L + \tau_G)(l - a) + \mu_1 \tau_L + \mu_3]\, s^3 \\ & + [l(b\tau_G + 1) - a + \mu_1 + \mu_2 \tau_L + \mu_3 b]\, s^2 \\ & + [b(l + \mu_1) + \mu_2]\, s + \mu_2 b. \end{aligned} \tag{3.6}$$

[8] Duesenberry's original form of investment function proposed is written as follows:

$$I_t = aY_{t-1} - bK_{t-1},$$

where t represents discrete-time.

4. Equilibrium Shifts

4.1. The case of the Kalecki-type investment function. This section investigates shifts in the equilibrium level caused by an exogenous demand shock and policy implementation in the case of the Kalecki-type investment function. If there exist stationary levels in the sense of convergence of $\lim_{t\to 0} Y(t)$, one can know such stationary levels based on the final value theorem (see Appendix), *i.e.*,

$$\lim_{s\to 0} s\mathcal{L}\left[Y(t)\right] = \lim_{t\to\infty} Y(t). \tag{4.1}$$

The following analyses of comparative statics focus on such possible stationary levels. We firstly show an influence of an exogenous change in autonomous demand on the stationary level in the economy without policy implementation.

PROPOSITION 4.1. *Let an economy be described by* (2.19)-(2.21) *and an exogenous shock in demand be* (2.22). *Without policy implementation* ($\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$)*, the stationary level of production shifts from* 0 *to* A/l*, if there exists a limiting value.*

Proof. Put $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$ in (2.21). Equations (2.19)–(2.21) become

$$\hat{Y}(s) = \frac{\tau_L s^2 + s + b}{l\tau_L(1-a)s^2 + l(1-a)s + lb}\hat{A}(s). \tag{4.2}$$

It follows from (2.22), (4.1) and (4.2) that

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s\hat{Y}(s) = \lim_{s\to 0} s\cdot \frac{\tau_L s^2 + s + b}{l\tau_L(1-a)s^2 + l(1-a)s + lb}\cdot\frac{A}{s} = \frac{A}{l}. \tag{4.3}$$

□

This proposition means that the initial stationary equilibrium level will not be restored naturally. The next proposition tells us the fact that the proportional policy restores the stationary level to some extent.

PROPOSITION 4.2. *Let an economy be described by* (2.19)-(2.21) *and an exogenous shock in demand be* (2.22). *The proportional policy* ($\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$) *can restore the stationary level of production to* $A/(l + \mu_1)$*, if there exists a limiting value.*

Proof. Put $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$ in (2.21). Equations (2.19)-(2.21) become

$$N(s) = \tau_L\tau_G s^3 + (\tau_L + \tau_G)s^2 + (1 + b\tau_G)s + b, \tag{4.4}$$

$$\begin{aligned} D(s) = {}& \left[\tau_L\tau_G l(1-a)\right]s^3 + \left[(\tau_L + \tau_G)l(1-a) + \mu_1\tau_L\right]s^2 \\ & + \left[l(b\tau_G + 1 - a) + \mu_1\right]s + b(l + \mu_1). \end{aligned} \tag{4.5}$$

Applying (2.22) and (4.1) to the above expression, we have

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s \cdot \frac{N(s)}{D(s)} \cdot \frac{A}{s} = \frac{A}{l+\mu_1}. \tag{4.6}$$

□

The next proposition asserts that production levels can be completely restored to the initial stationary level through the Phillips-type policy.

PROPOSITION 4.3. *Let an economy be described by* (2.19)-(2.21) *and an exogenous shock in demand be* (2.22). *The Phillips-type policy* ($\mu_1 > 0,\ \mu_2 > 0,\ \mu_3 > 0$) *can restore the stationary level of production to* 0, *if there exists a limiting value.*

In fact, from (2.19)-(2.22) and (4.1), we obtain

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s \cdot \frac{N(s)}{D(s)} \cdot \frac{A}{s} = 0. \tag{4.7}$$

The stationary levels found in Proposition 4.1- 4.3 are the same as those in the original form of Phillips' model. We find that the existence of capital accumulation does not affect the stationary level and the Phillips-type policy is effective in offsetting a shift in demand even with capital accumulation.

It is now easy to see that in the cases of $\mu_1 = 0,\ \mu_2 > 0,\ \mu_3 = 0$ and $\mu_1 = 0,\ \mu_2 = 0,\ \mu_3 > 0$, the stationary equilibrium levels are 0 and A/l, respectively. Namely, with the integral policy, the equilibrium position can be completely restored. The derivative policy cannot affect the equilibrium level, but it is introduced with the object of suppressing oscillations caused by introducing the integral policy which increases the order of differential equations and consequently causes oscillations.

4.2. The case of the Duesenberry-type investment function. Using the arguments of the previous section, we analyze equilibrium shifts in the case of the Duesenberry-type investment function. The following propositions tell us that behaviors of equilibrium shifts under the Duesenberry-type are the same as those under the Kalecki-type.

PROPOSITION 4.4. *Let an economy be described by* (3.4)-(3.6) *and an exogenous shock in demand be* (2.22). *Without policy implementation* ($\mu_1 = 0,\ \mu_2 = 0,\ \mu_3 = 0$), *the stationary level of production shifts from* 0 *to* A/l, *if there exists a limiting value.*

Proof. Put $\mu_1 = 0,\ \mu_2 = 0,\ \mu_3 = 0$ in (3.6). Equations (3.4)-(3.6) become

$$\hat{Y}(s) = \frac{\tau_L s^2 + s + b}{\tau_L(l-a)s^2 + (l-a)s + lb}\hat{A}(s). \tag{4.8}$$

Thus, it follows from (2.22), (4.1), (4.8) that

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s\hat{Y}(s) = \lim_{s\to 0} s\cdot\frac{\tau_L s^2 + s + b}{\tau_L(l-a)s^2 + (l-a)s + lb}\cdot\frac{A}{s} = \frac{A}{l}. \tag{4.9}$$

□

In any combination of proportional, integral, and derivative terms, we obtain the same equilibrium shifts as those on the Kalecki-type investment function. The next proposition shows that the Phillips-type policy has a performance of offsetting an equilibrium shift caused by an exogenous demand shock.

PROPOSITION 4.5. *Let an economy be described by* (3.4)-(3.6) *and an exogenous shock in demand be* (2.22). *The Phillips-type policy* ($\mu_1 > 0,\ \mu_2 > 0,\ \mu_3 > 0$) *can restore the stationary level of production to* 0, *if there exists a limiting value.*

In fact, by applying (4.1) to (2.22) and (3.4)-(3.6), we obtain

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s\cdot\frac{N(s)}{D(s)}\cdot\frac{A}{s} = 0. \tag{4.10}$$

In the other cases, we can obtain the same stationary levels as those in the case of the Kalecki-type investment function.

5. Stability Conditions

5.1. The case of the Kalecki-type investment function. Since Proposition 4.1-4.5 assume the existence of $\lim_{t\to\infty} Y(t)$, we must show conditions to ensure the convergence. This section is devoted to deriving a necessary and sufficient condition for asymptotic stability in the case of the Kalecki-type investment function.

PROPOSITION 5.1. *Let an economy be described by* (2.19)-(2.21). *A necessary and sufficient condition for asymptotic stability of the economy is that all the following inequalities are satisfied,*

$$\begin{aligned} a &< 1 + \frac{\mu_3}{\tau_G l}, \\ a &< 1 + \frac{\mu_1\tau_L + \mu_3}{(\tau_L + \tau_G)l}, \\ a &< 1 + b\tau_G + \frac{\mu_1 + \mu_2\tau_L + \mu_3 b}{l}, \end{aligned} \tag{5.1}$$

and all the leading principal minors of the following matrix are positive:

$$\begin{pmatrix} \kappa_1 & b(l+\mu_1)+\mu_2 & 0 & 0 \\ \kappa_2 & \kappa_3 & \mu_2 b & 0 \\ 0 & \kappa_1 & b(l+\mu_1)+\mu_2 & 0 \\ 0 & \kappa_2 & \kappa_3 & \mu_2 b \end{pmatrix}, \tag{5.2}$$

where $\kappa_1 = (\tau_L + \tau_G)l(1-a) + \mu_1\tau_L + \mu_3$, $\kappa_2 = \tau_L\tau_G l(1-a) + \mu_3\tau_L$ *and* $\kappa_3 = l(b\tau_G + 1 - a) + \mu_1 + \mu_2\tau_L + \mu_3 b$.[9]

Proof. Applying the Routh-Hurwitz theorem to the denominator polynomial (2.21), one obtains a necessary and sufficient condition for all the roots of (2.21) to have negative real parts. This condition is equivalent to a necessary and sufficient condition for stability, because the roots of denominator polynomials are exactly characteristic roots of differential equations.[10] The Routh-Hurwitz theorem requires that the coefficients of (2.21) exist and have the same sign. Since the conditions $b(l+\mu_1)+\mu_2 > 0$ and $\mu_2 b > 0$ are always satisfied by the definition, these conditions have no significance. Thus, the other coefficients must be positive, *i.e.*, $\tau_L\tau_G l(1-a)+\mu_3\tau_L > 0$, $(\tau_L+\tau_G)l(1-a)+\mu_1\tau_L+\mu_3 > 0$, $l(b\tau_G+1-a)+\mu_1+\mu_2\tau_L+\mu_3 b > 0$. These conditions eventually become (5.1).

In addition, all the leading principal minors of the Hurwitz matrix of (2.21) must be positive. Generally, the Hurwitz matrix of $c_4s^4 + c_3s^3 + c_2s^2 + c_1s + c_0$ is described as

$$\begin{pmatrix} c_3 & c_1 & 0 & 0 \\ c_4 & c_2 & c_0 & 0 \\ 0 & c_3 & c_1 & 0 \\ 0 & c_4 & c_2 & c_0 \end{pmatrix}.$$

The matrix (5.2) is the corresponding Hurwitz matrix of the polynomial (2.21). □

The above stability condition indicates a trade-off between parameters of the policy and the economic structure. As can be seen from (5.1), the value of a is required to be small to some extent for stability. In contrast, increasing the value of b loosens the constraint imposed on that of a.

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[9] Namely,

$$c_{n-1} > 0, \quad \begin{vmatrix} c_{n-1} & c_{n-3} \\ c_n & c_{n-2} \end{vmatrix} > 0, \quad \begin{vmatrix} c_{n-1} & c_{n-3} & c_{n-5} \\ c_n & c_{n-2} & c_{n-4} \\ 0 & c_{n-1} & c_{n-3} \end{vmatrix} > 0, \quad \cdots.$$

[10] See, *e.g.*, Hurwitz (1895), Gantmacher (1959), mathematical appendix B in Samuelson (1947), and Bakshi and Bakshi (2010).

We consider the influence of policy implementation on stability. Let $\mu_1,\ \mu_2$ and $\mu_3 \to 0$ in (5.1), and we find that the conditions (5.1) converge to a condition $a < 1$. Moreover, the condition on the leading principal minors of (5.2) is written as follows:

$$
\begin{aligned}
& a < 1, \\
& a < 1 + b\tau_G \left(1 - \frac{\tau_L}{\tau_L + \tau_G}\right).
\end{aligned}
\tag{5.3}
$$

Note that the leading principal minor of the matrix with order 4 has the same property as the leading principal minor with order 3. From the assumption that $\tau_L,\ \tau_G > 0$, we have

$$1 - \frac{\tau_L}{\tau_L + \tau_G} > 0.$$

Thus, the conditions (5.3) become $a < 1$. It turns out that on the Kalecki-type investment function, the necessary and sufficient condition is $a < 1$ when $\mu_1,\ \mu_2$ and $\mu_3 \to 0$. Interestingly, this condition is also found in the original form of Kalecki's business cycle model. Lange (1970) argued that a necessary condition for stability in Kalecki's model is $a < 1$ (hereinafter referred to as the Kaleckian stability condition). Namely, in our model, the Kaleckian stability condition is relegated to a limiting case where policy parameters are small enough to be ignored. In Kalecki's model, there is no expanded reproduction and full reproduction leads to instability. Lange considers this situation to be the basic dilemma in the capitalist economy: growth and instability, or stability and stagnation or even decline. In (2.5), if $K(t) > 0$, the stability condition $a < 1$ directly results in a relation $I(t) < lY(t)$. This economy is stable but full reproduction is impossible. On the other hand, if we take $a \geq 1$ in order to increase the amount of investment, the economy becomes unstable owing to the stability condition. This fact indicates that without policy there is a basic incompatibility between full reproduction and stability in both Kalecki's model and our model. In our model, when policy implementation is small enough to be ignored, such a dilemma remains. However, our model indicates that when the values of policy parameters are sufficiently large, the condition $a \geq 1$ does not necessarily lead to instability. The economy can stably restore the initial stationary level through the Phillips-type policy, allowing that $a \geq 1$. Lange's dilemma is resolved in this sense.

As can be seen from (5.1), increasing the value of l strengthens the constraint on that of a. This feature reflects the fact that the amount of investment decisions is assumed to depend on that of savings with the term $alY(t)$.

5.2. The case of the Duesenberry-type investment function. This section considers the stability condition on the Duesenberry-type investment function.

PROPOSITION 5.2. *Let an economy be described by* (3.4)-(3.6)*. A necessary and sufficient condition for asymptotic stability of the economy is that all the following inequalities are satisfied,*

$$
\begin{aligned}
&a < l + \frac{\mu_3}{\tau_G l},\\
&a < l + \frac{\mu_1 \tau_L + \mu_3}{(\tau_L + \tau_G) l},\\
&a < l(1 + b\tau_G) + \mu_1 + \mu_2 \tau_L + \mu_3 b,
\end{aligned}
\tag{5.4}
$$

and all the leading principal minors of the following matrix are positive:

$$
\begin{pmatrix}
\lambda_1 & b(l+\mu_1)+\mu_2 & 0 & 0 \\
\tau_L \tau_G (l-a) + \mu_3 \tau_L & \lambda_2 & \mu_2 b & 0 \\
0 & \lambda_1 & b(l+\mu_1)+\mu_2 & 0 \\
0 & \tau_L \tau_G (l-a) + \mu_3 \tau_L & \lambda_2 & \mu_2 b
\end{pmatrix},
\tag{5.5}
$$

where $\lambda_1 = (\tau_L + \tau_G)(l - a) + \mu_1 \tau_L + \mu_3$, $\lambda_2 = l(b\tau_G + 1) - a + \mu_1 + \mu_2 \tau_L + \mu_3 b$.

In fact, we can apply the argument of Proposition 5.1 to (3.6).

In contrast to the case of the Kalecki-type, increasing the value of l loosens the constraint on that of a in (5.4). This feature comes from the fact that the Duesenberry-type investment function depends on $aY(t)$, not on $alY(t)$. It is also shown that a condition $a < l$ is a special case obtained by letting μ_1, μ_2 and $\mu_3 \to 0$. Suppose that μ_1, μ_2 and $\mu_3 \to 0$. The condition on the leading principal minors of (5.5) is written as follows:

$$
\begin{aligned}
&a < l,\\
&a < l + lb\tau_G \left(1 - \frac{\tau_L}{\tau_L + \tau_G}\right).
\end{aligned}
\tag{5.6}
$$

With the same argument in the previous section, we see that the necessary and sufficient condition eventually becomes $a < l$.

The so-called Keynesian condition, which is imposed on Kaldorian macrodynamic models, means that the saving rate is greater than the sensitivity of investment to production level. In our model, since a is the sensitivity of investment to production level, our condition $a < l$ is interpreted as the Keynesian stability condition. As with the Kaleckian stability condition, the Keynesian stability condition immediately results in the impossibility of full reproduction. In (3.2), if $K(t) > 0$, the condition $a < l$ directly gives $I(t) < aY(t) < lY(t)$; if we take $a \geq l$, the economy becomes unstable. However, even when $a \geq l$, the economy can be stabilized by means of the Phillips-type policy. Thus, we also see that the Keynesian stability condition has significance for stability only in the case without policies.

We now give a numerical example. Assume that $l = 0.5$, $a = 0.5$, $b = 0.1$, $\tau_L = 1$, $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$. This case does not satisfy the Keynesian stability condition but we can demonstrate that this economy is stable. First, the condition (5.4) is clearly satisfied from the definition. Next, the Hurwitz matrix of this numerical example is given by

$$\begin{pmatrix} 2 & 1.15 & 0 & 0 \\ 1 & 0.05\tau_G + 2.1 & 0.1 & 0 \\ 0 & 2 & 1.15 & 0 \\ 0 & 1 & 0.05\tau_G + 2.1 & 0.1 \end{pmatrix}.$$

It is easy to see that all the leading principal minors of this matrix are positive regardless of τ_G. This case does not satisfy the Keynesian stability condition; nevertheless the economy is stable owing to policy implementation.

5.3. Effects of policy lag on stability. This section shows that there are some cases where an economy can be stabilized regardless of the length of policy lags. Friedman (1948) intuitively pointed out the possibility of destabilization due to the existence of policy lags. Indeed, our stability conditions imply that the value of τ_G can be a destabilizing factor. However, an appropriate selection of policy parameters can compensate the destabilizing effect of τ_G. Namely, the value of τ_G does not generally cause instability. Asada and Yoshida (2000) introduced a policy with proportional and constant terms into a dynamic Keynesian IS-LM model and showed that too long lags of policy implementation fail to stabilize the economy. However, there are cases where τ_G has no destabilizing effect.

Let us consider a sufficient condition for stability when $a = 1$ in the Kalecki-type, or when $a = l$ in the Duesenberry-type investment function.

PROPOSITION 5.3. *Let an economy be described by* (2.19)-(2.21) (*or by* (3.4)-(3.6)). *Suppose that* $a = 1$ (*or* $a = l$). *We normalize the length of capital goods production lag by putting* $\tau_L = 1$. *A sufficient condition for asymptotic stability of the economy is that all the following inequalities are satisfied*:

$$b < \frac{1}{2}, \quad l < \mu_3, \quad \mu_2 < \mu_1. \tag{5.7}$$

Proof. The above conditions satisfy the condition (5.1) when $a = 1$ (or (5.4) when $a = l$). From (5.2) (or (5.5)), we obtain the following Hurwitz matrix:

$$\begin{pmatrix} \mu_1 + \mu_3 & b(l + \mu_1) + \mu_2 & 0 & 0 \\ \mu_3 & bl\tau_G + \mu_1 + \mu_2 + \mu_3 b & \mu_2 b & 0 \\ 0 & \mu_1 + \mu_3 & b(l + \mu_1) + \mu_2 & 0 \\ 0 & \mu_3 & bl\tau_G + \mu_1 + \mu_2 + \mu_3 b & \mu_2 b \end{pmatrix}.$$

The conditions on the leading principal minors of this matrix are given by

$$\mu_1 + \mu_3 > 0, \tag{5.8}$$

$$\begin{vmatrix} \mu_1 + \mu_3 & b(l + \mu_1) + \mu_2 \\ \mu_3 & bl\tau_G + \mu_1 + \mu_2 + \mu_3 b \end{vmatrix} > 0, \tag{5.9}$$

$$\begin{vmatrix} \mu_1 + \mu_3 & b(l + \mu_1) + \mu_2 & 0 \\ \mu_3 & bl\tau_G + \mu_1 + \mu_2 + \mu_3 b & \mu_2 b \\ 0 & \mu_1 + \mu_3 & b(l + \mu_1) + \mu_2 \end{vmatrix} > 0. \tag{5.10}$$

The leading principal minor of the matrix with order 4 has the same property as the leading principal minor with order 3. Then, the above relations become

$$\mu_1(\mu_1 + \mu_2 + \mu_3) + b\tau_G l(\mu_1 + \mu_3) + b\mu_3(\mu_3 - l) > 0, \tag{5.11}$$

$$\begin{aligned} & b^2\tau_G l^2(\mu_1 + \mu_3) + b^2\tau_G l(\mu_1^2 + \mu_1\mu_3) + b^2\mu_3(\mu_1 + l)(\mu_3 - l) \\ & \quad + b\tau_G l(\mu_1\mu_2 + \mu_2\mu_3) + bl(\mu_1^2 + \mu_1\mu_2) + bl\mu_3(\mu_1 - \mu_2) \\ & \qquad + b\mu_1(\mu_1^2 + \mu_1\mu_3) + \mu_1\mu_2(\mu_1 + \mu_2) + \mu_1\mu_2\mu_3(1 - 2b) > 0. \end{aligned} \tag{5.12}$$

If inequalities $\mu_3 - l > 0$, $\mu_1 - \mu_2 > 0$, $1 - 2b > 0$ are satisfied, the conditions (5.11) and (5.12) are always satisfied. □

In the above case, it is evident that the value of τ_G does not affect whether the stability conditions (5.11) and (5.12) are satisfied or not. This means that the length of policy lags has no destabilizing effect in this case.

In reality, the above constraint of the value of b is not strict, because b takes quite small values. A statistical estimation of the value of b by Kalecki (1935) indicates $b = 0.12$. Moreover, the effect of increasing the value of b is easily offset. Therefore, even when $b > \frac{1}{2}$, the Phillips-type policy has wide applicability.

6. Conclusions

We have studied stabilization effects of the Phillips-type policy in economies with capital accumulation, comparing the cases of investment functions of Kalecki-type and Duesenberry-type. Within our framework, the existence of capital accumulation does not affect behaviors of the equilibrium shifts and the Phillips-type policy still has the stabilization effect. The difference between ideas of Kaleckian and Keynesian on investment decisions is reflected in the stability conditions, particularly as regards effects of the saving rate. The stability conditions tell us that the Kaleckian stability condition and the Keynesian stability condition have significance only when the amounts of policy implementation can be ignored. Even when these conditions are not satisfied, the economy can be stabilized by the Phillips-type policy.

Lange's dilemma is resolved in the sense of compatibility between full reproduction and stabilization. Moreover, we found that a time-lag of policy implementation is not generally a destabilizing factor.

Future work in this area will require the consideration of financial circumstances of policy instruments and investments, but our results will give an insight into stabilization problems in economies accompanied not only by full reproduction but also by more investment through the credit creation beyond savings.

Appendix

The Laplace transforms of derivatives and integrals. The Laplace transforms of the derivatives of $f(t)$ are given by the following relation:

$$\text{(A.1)}\quad \mathcal{L}\left[f^{(n)}(t)\right] = s^n\mathcal{L}\left[f(t)\right] - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-),$$

where we assumed that $f(t),\ f'(t),\ \cdots,\ f^{(n-1)}(t)$ are continuous on $(0,\infty)$ and of exponential order and that $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order. $f(0^-), f'(0^-), \cdots$, and $f^{(n-1)}(0^-)$ denote each of the initial values as the left-hand limit at the origin. Thus, we obtain

$$\text{(A.2)}\qquad \mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}\left[f(t)\right]$$

with the initial condition $f(0^-) = f'(0^-) = \cdots f^{(n-1)}(0^-) = 0$.

Put $g(t) = \int_0^t f(\tau)d\tau$, and the Laplace transform of the integral of $f(t)$ is described as follows:

$$\text{(A.3)}\qquad \mathcal{L}\left[f(t)\right] = \mathcal{L}\left[g'(t)\right] = s\mathcal{L}\left[g(t)\right] - g(0^-) = s\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]$$

where $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order.

The final value theorem. Let us mention the final value theorem (see, *e.g.*, Simon (1952) and Kreyszig (2011)). Suppose that $\lim_{t\to\infty} f(t)$ exists. Consider the following relation:

$$\lim_{s\to 0}\mathcal{L}\left[f'(t)\right] = \lim_{s\to 0}\int_{0^-}^{\infty} e^{-st} f'(t)dt = \int_{0^-}^{\infty} f'(t)dt = \lim_{b\to\infty}\left[f(b) - f(0^-)\right].$$

From $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0^-)$, we obtain the final value theorem:

$$\lim_{s\to 0} s\mathcal{L}[f(t)] = \lim_{t\to\infty} f(t) \tag{A.4}$$

with the initial condition $f(0^-) = 0$.

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