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Stability in Macrodynamic Models with a Policy Lag: Old- vs New Keynesian Economics

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ABSTRACT. In this study, we examine the effects of a monetary policy lag in the standard Old and New Keynesian macrodynamic models, and compare their respective policy implications concerning macroeconomic stability. An Old Keynesian model derives the following policy norm: a monetary policy lag may cause economic instability. In contrast, a New Keynesian model derives the following norm: a moderate policy lag can contribute to stabilizing an economy. This difference in policy implications can be attributed to the difference between the stability concepts in Old and New Keynesian economics, *i.e.*, the difference between the traditional stability concept and the concept of equilibrium determinacy.

***JEL Classification*:** E12; E30; E62

***Keywords*:** monetary policy rule; system of delay differential equations; policy lag; stability analysis

1. Introduction

The New Keynesian (NK) model has attracted strong criticism. In particular, Fujimori(2009), Asada(2013), and Tsuduki(2014) argue that the peculiarity of its stability concept is one of the most severe problems of this model.

In the traditional mathematics and physics sense, if the real parts of all eigenvalues of a continuous-time dynamic system are negative, the steady state is considered as locally stable. Post-Keynesian scholars apply this standard concept to the Kaldorian business cycle model, which is one of the most popular Post-Keynesian models, to derive the following economic implication: For economic stability, the marginal propensity to save should be greater than the marginal propensity to invest.

NK scholars use a different stability concept. First, they distinguish between two types of economic variables: the jump variable and the non-jump variable. Concerning a jump variable, the initial value of a variable is not given but can be

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determined by economic agents, typically by households. In contrast, a non-jump variable is one whose initial value is historically given and cannot be reset by agents.

Based on this discrimination, NK scholars apply the concept of equilibrium determinacy. If a path that converges to the steady state uniquely exists, the equilibrium is considered as determinate. If multiple equilibrium paths exist, however, equilibrium is indeterminate. Moreover, NK scholars regard such a multiplicity of equilibrium paths as one of the factors causing the economic fluctuation that is driven by volatile expectations, often known as sunspot fluctuations. Thus, equilibrium indeterminacy may cause economic instability.

For example, let us consider a two-dimensional system of differential equations wherein both endogenous variables are jump variables, which is actually the case in the standard NK model. If the system has at least one eigenvalue with negative real parts, equilibrium is indeterminate; if both eigenvalues have positive real parts, the equilibrium is determinate; only the steady-state equilibrium path exists.

To the traditional angle, the concept of equilibrium determinacy is very strange. This is because, practically, the case of a determinate equilibrium should be regarded unstable; *i.e.*, it is almost impossible for economic agents to place the initial values of the jump variables exactly in the steady state.

We examine macrodynamic models with a policy lag, *i.e.*, a delay in a central bank's policy response, to examine the effects of monetary policy on economic stability. Comparing the policy implications derived from the traditional Keynesian model with those derived from the NK model, we show that the NK stability concept yields a peculiar policy implication.

2. Old Keynesian Model

2.1. Case of no policy lag. Asada and Yoshida(2001) and Yoshida and Asada(2007) examine the effect of a fiscal policy lag on economic stability using the Keynes-Goodwin model and the Kaldor model, respectively. Asada(2010, 2012) examines the stabilizing effect of a monetary policy using traditional Keynesian macrodynamic models. With modesty and some pride, Asada terms these models the "Old Keynesian" (OK) models.

Asada(2010, 2012) emphasizes the importance of the credibility of the central bank with regard to economic stability. If private economic agents have sufficiently forward-looking inflationary expectations (in other words, the central bank is sufficiently credible), an economy will be stable. However, if these agents have backward-looking inflationary expectations (in other words, the expectation formations are adaptive), an economy will become unstable.

This section re-examines the effects of a monetary policy by introducing a policy lag into the simplest OK model proposed by Asada(2010). First, we state the original model without a policy lag.

2.1.1. ***Foundational equations.*** The model economy comprises the following equations:

$$Y = Y(R - \pi^e); \quad Y' < 0, \tag{2.1}$$

$$\frac{M}{P} = L(Y, R, \pi^e); \quad L_Y > 0;\ L_R < 0;\ L_{\pi^e} < 0, \tag{2.2}$$

$$\pi = \varepsilon(Y - \bar{Y}) + \pi^e, \tag{2.3}$$

$$\dot{R} = \omega(\pi - \bar{\pi}), \tag{2.4}$$

$$\dot{\pi}^e = \delta[\tau(\bar{\pi} - \pi^e) + (1 - \tau)(\pi - \pi^e)], \tag{2.5}$$

where Y = real national income; R = nominal interest rate; π^e = expected inflation rate; M = nominal money supply; P = general price level; $\pi = \dot{P}/P$ = inflation rate; $\bar{Y}$ = natural level of the real national income > 0; $\bar{\pi}$ = target inflation rate level > 0; $\varepsilon > 0$; $\omega > 0$; $\delta > 0$; and $0 < \tau < 1$.

(2.1) is the equilibrium condition for the goods market, *i.e.*, IS curve; (2.2) is the equilibrium condition for the monetary market, *i.e.*, LM curve. (2.3) is the expectations-augmented Phillips curve; and (2.4) is a monetary policy rule wherein the nominal interest rate varies in response to changes in the inflation rate.[1] Asada(2010) considers the zero lower bound constraint on the nominal interest rate, whereas we simply assume that the nominal interest rate is always positive, which implies that we only examine the normal condition of an economy.

(2.5) represents the formation of inflationary expectations. The first term in parenthesis shows that when private agents form their expectations, they believe that the current inflation rate will converge to the target level. Therefore, this term represents a forward-looking expectation formation. In contrast, the second term in the parenthesis represents a backward-looking, *i.e.*, adaptive, expectation formation. Thus, the parameter τ can be considered as reflecting the credibility of the central bank or the proportion of backward-looking people.

[1] Asada(2010) assumes that the nominal interest rate responds not only to the inflation rate but also to the real national income; however, we omit the latter part for simplicity. This alteration does not affect our main conclusion.

Substituting (2.1) and (2.3) into (2.4) and (2.5), we obtain the following system of differential equations:

$$\begin{aligned} \dot{R} &= \omega[\varepsilon\{Y(R-\pi^e)-\bar{Y}\}+\pi^e-\bar{\pi}], \\ \dot{\pi}^e &= \delta[\tau(\bar{\pi}-\pi^e)+(1-\tau)\varepsilon\{Y(R-\pi^e)-\bar{Y}\}]. \end{aligned} \tag{2.6}$$

These equations determine the dynamics of R and π^e, and, hence, Y. Given these dynamics, (2.2) determines the dynamics of M. Thus, the economic system described by (2.1) to (2.5) is decomposable. We should only examine the dynamics of (2.6) to reveal the local stability of the model economy.

2.1.2. *Local stability.* The steady-state values of (2.6) are given as follows:

$$\begin{aligned} Y(R^* - \bar{\pi}) &= \bar{Y}, \\ \pi^{e*} &= \bar{\pi}. \end{aligned} \tag{2.7}$$

The Jacobian matrix of (2.6), evaluated at the steady state, can be given as follows:

$$J_1 = \begin{bmatrix} \omega\varepsilon Y'^* & -\omega\varepsilon Y'^* + \omega \\ \delta(1-\tau)\varepsilon Y'^* & \delta\{-\tau-(1-\tau)\varepsilon Y'^*\} \end{bmatrix}.$$

Thus, the characteristic equation of this system can be represented as follows:

$$\Delta_1(\lambda) \equiv |\lambda I - J_1| = \lambda^2 + a_1\lambda + a_2 = 0, \tag{2.8}$$

where $a_1 = -\text{tr}\, J_1 = -\varepsilon\{\omega-(1-\tau)\delta\}Y'^* + \delta\tau,\ a_2 = \det J_1 = -\delta\omega\varepsilon Y'^* > 0$.

We focus on the political parameter ω, which represents the reactivity of the nominal interest rate to the inflation rate. Defining $\omega_0 \equiv \dfrac{\delta\tau}{\varepsilon Y'^*} + (1-\tau)\delta$, we can derive the following relationships:

$$\omega \gtreqless \omega_0 \qquad \Longleftrightarrow \qquad \text{tr}\, J_1 \lesseqgtr 0.$$

Furthermore, noting that the value of the trace generally equals the sum of the roots and the value of the determinant equals the product of the roots, we can establish the following proposition:

PROPOSITION 2.1. *If $\omega > \omega_0$, the steady state is a sink. Hence, it is stable. However, if $\omega < \omega_0$, the steady state is a source and, therefore, unstable.*

This proposition suggests that if the central bank is sufficiently "active," an economy will be stable.

In addition, we can also develop the same argument for parameter τ. If the value of τ is sufficiently large, which implies that the central bank is very credible, then tr $J_1 < 0$, and accordingly, the steady state is stable.

Incidentally, as demonstrated by Asada(2010), ω_0 is a Hopf bifurcation point. Hence, non-steady cycles exist for certain values of ω in the neighborhood of ω_0.

2.2. Case of a positive policy lag.

2.2.1. *Delay-differential equations.* In the case of a policy with a lag, the policy rule in (2.4) can be rewritten as follows:

$$\dot{R}(t) = \omega(\pi(t-\theta) - \bar{\pi}). \tag{2.9}$$

This equation can be considered as depicting the situation where a delay of $\theta > 0$ exists in the central bank's policy response to economic fluctuations.[2]

Replacing (2.9) with (2.4), the model economy can, then, be represented by the following system of delay-differential equations:

$$\begin{aligned} \dot{R}(t) &= \omega[\varepsilon\{Y(R(t-\theta) - \pi^e(t-\theta)) - \bar{Y}\} + \pi^e(t-\theta) - \bar{\pi}], \\ \dot{\pi}^e(t) &= \delta[\tau(\bar{\pi} - \pi^e(t)) + (1-\tau)\varepsilon\{Y(R(t) - \pi^e(t)) - \bar{Y}\}]. \end{aligned} \tag{2.10}$$

The steady-state values of (2.10) are given by (2.7). To analyze the local dynamics, we linearize (2.10) around the steady state, and obtain

$$\begin{aligned} \dot{\hat{R}}(t) &= \omega[\varepsilon Y'^{*}\hat{R}(t-\theta) + (1-\varepsilon Y'^{*})\hat{\pi}^e(t-\theta)], \\ \dot{\hat{\pi}}^e(t) &= \delta[\{-\tau - (1-\tau)\varepsilon Y'^{*}\}\hat{\pi}^e(t) + (1-\tau)\varepsilon Y'^{*}\hat{R}(t)], \end{aligned} \tag{2.11}$$

where $\hat{R}(t) \equiv R(t) - R^*$, and $\hat{\pi}^e(t) \equiv \pi^e(t) - \pi^{e*}$.

Assuming the exponential functions $\hat{R}(t) = C_1 e^{\lambda t}$ and $\hat{\pi}^e(t) = C_2 e^{\lambda t}$ as the solutions of this system, and substituting these into (2.11), we obtain

$$\begin{bmatrix} \lambda - \omega\varepsilon Y'^{*} e^{-\theta\lambda} & -\omega(1-\varepsilon Y'^{*})e^{-\theta\lambda} \\ -\delta(1-\tau)\varepsilon Y'^{*} & \lambda - \delta\{-\tau - (1-\tau)\varepsilon Y'^{*}\} \end{bmatrix} \begin{bmatrix} \hat{R}(t) \\ \hat{\pi}^e(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For non-trivial solutions to exist, the determinant of the left-hand side matrix must equal zero; that is

$$\Delta_2(\lambda) \equiv \lambda^2 - \delta\{-\tau - (1-\tau)\varepsilon Y'^{*}\}\lambda - \omega\varepsilon Y'^{*}(\lambda + \delta)e^{-\theta\lambda} = 0, \tag{2.12}$$

which is the characteristic equation of (2.10).

(2.12) has infinite solutions. This is due to the existence of the term with the exponential function $e^{-\theta\lambda}$ (Bellman and Cooke (1963, Ch. 3).) Therefore, if at

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[2] Asada and Matsumoto(2014) analyze the case wherein delay is not constant but distributed:

$$R(t) = R^* + \omega[(1/\theta)\int_{-\infty}^{t} e^{-(1/\theta)(t-s)}\pi(s)ds - \pi^*]$$

.

least one of the infinite number of roots has positive real parts, the steady state is unstable. However, if all roots have negative real parts, the steady state is stable.

To observe the effect of lag time on stability, we assume the following:

ASSUMPTION 2.1. $\omega > \omega_0$.

The procedures of the analysis follow Matsumoto and Szidarovszky (2012).

(1) We characterize the points, if any, at which the dynamics around the steady state can change, *i.e.*, the points at which a zero real root or the pure imaginary roots appear. These points are referred to as "crossing points."

(2) We reveal the directions of the changes in the signs of the real parts that occur when θ crosses the crossing points.

2.2.2. *Crossing points.* First, we examine whether the points at which a zero real root appears can exist. By substituting $\lambda = 0$ into (2.12), we obtain $\Delta_2(0) = -\omega\varepsilon\delta Y'^* = 0$. This equality cannot hold. Accordingly, $\lambda = 0$ cannot be a root.

Next, we examine whether the points at which the pure imaginary roots appear can exist. By substituting $\lambda = iz$ (z = imaginary part > 0;[3] $i = \sqrt{-1}$) into (2.12), we obtain

$$-z^2 - \delta\{-\tau - (1-\tau)\varepsilon Y'^*\}zi - \omega\varepsilon Y'^*(zi + \delta)e^{-\theta zi} = 0.$$

Application of Euler's formula ($\forall x \in \mathbb{R}$, $e^{\pm ix} = \cos x \pm i \sin x$) to this equation yields

$$\begin{aligned} &-z^2 - \omega\varepsilon Y'^* z \sin\theta z - \omega\varepsilon Y'^*\delta\cos\theta z \\ &\quad - i[\delta\{-\tau - (1-\tau)\varepsilon Y'^*\}z + \omega\varepsilon Y'^* z\cos\theta z - \omega\varepsilon Y'^*\delta\sin\theta z] = 0. \end{aligned}$$

Both the real and imaginary parts of the left-hand side of this expression must be zero for the equality to hold here; *i.e.*,

$$z^2 + \omega\varepsilon Y'^* z\sin\theta z + \omega\varepsilon Y'^*\delta\cos\theta z = 0, \tag{2.13}$$

$$\delta\{-\tau - (1-\tau)\varepsilon Y'^*\}z + \omega\varepsilon Y'^* z\cos\theta z - \omega\varepsilon Y'^*\delta\sin\theta z = 0. \tag{2.14}$$

Solving (2.13) and (2.14) for $\cos\theta z$, we obtain

$$\cos\theta z = \frac{\delta z^2\{\tau + (1-\tau)\varepsilon Y'^* - 1\}}{\omega\varepsilon Y'^*(\delta^2 + z^2)}, \tag{2.15}$$

where $2\pi h < \theta z < 2\pi(1+h)$, $h = 0, 1, 2, 3, \cdots$ and $\pi = 3.14159\cdots$.

[3] Pure imaginary roots are always conjugated. Therefore, we can assume $z > 0$ without a loss of generality.

Furthermore, the sum of the square of (2.13) and (2.14) yields the following fourth degree equation of z:

$$z^4 + [\delta^2\{-\tau - (1-\tau)\varepsilon Y'^*\}^2 - \omega^2\varepsilon^2(Y'^*)^2]z^2 - \omega^2\varepsilon^2(Y'^*)^2\delta^2 = 0.$$

Solving for z^2, we obtain

$$z_\pm^2 = \frac{-[\delta^2\{-\tau - (1-\tau)\varepsilon Y'^*\}^2 - \omega^2\varepsilon^2(Y'^*)^2] \pm \sqrt{N}}{2}, \tag{2.16}$$

where $N \equiv [\delta^2\{-\tau - (1-\tau)\varepsilon Y'^*\}^2 - \omega^2\varepsilon^2(Y'^*)^2]^2 + 4\omega^2\varepsilon^2(Y'^*)^2\delta^2 > 0$.

(2.16) demonstrates that $z_+^2 > 0$ and $z_-^2 < 0$. By substituting a real and positive value of z (*i.e.*, z_+) into (2.15) and solving for θ, we obtain

(2.17)

$$\theta_h = \frac{1}{z_+}\cos^{-1}\left[\frac{\delta z_+^2\{\tau + (1-\tau)\varepsilon Y'^* - 1\}}{\omega\varepsilon Y'^*(\delta^2 + z_+^2)}\right] + \frac{2\pi h}{z_+}, \quad h = 0, 1, 2, 3, \cdots.$$

Thus, there exists an infinite number of values of θ (*i.e.*, $\theta_0, \theta_1, \theta_2, \cdots$) that generate pure imaginary roots, *i.e.*, crossing points.

2.2.3. *Direction of crossing.* In this subsection, we reveal the direction of the changes in the signs of the complex roots that occur when θ crosses θ_h. This is determined by the sign of $d\text{Re}\lambda/d\theta|_{\lambda=iz_+}$. If $d\text{Re}\lambda/d\theta|_{\lambda=iz_+} > 0$ for any value of h, the signs of the real parts of the roots always change from negative to positive with an increase in θ. That is, the number of roots with positive real parts increases by two each time there is a crossing.

For convenience of calculation, we observe the sign of $\text{Re}(d\lambda/d\theta)^{-1}|_{\lambda=iz_+}$ instead of that of $d\text{Re}\lambda/d\theta|_{\lambda=iz_+}$. (2.12) demonstrates that the following inequality holds for all h (see Appendix A.1):

$$\text{Re}\left(\frac{d\lambda}{d\theta}\right)^{-1}\Bigg|_{\lambda=iz_+} > 0.$$

Therefore, with any crossing, the signs of the real parts of the roots change from negative to positive.

Incidentally, as demonstrated by Proposition 2.1, if $\theta = 0$, roots with positive real parts do not exist. Moreover, with increases in θ, the number of roots with positive real parts increases by two. Accordingly, we can state that $2h$ roots with positive real parts exist in the region of $\theta \in (\theta_{h-1}, \theta_h)$, where $\theta_{-1} = 0$.

Thus, we can establish the following proposition:

PROPOSITION 2.2. *Under Assumption 2.1, if $\theta \in (0, \theta_0)$, the steady state is locally stable. However, if $\theta \in (\theta_0, \infty)$, the steady state is unstable.*

This proposition suggests that even if the central bank is sufficiently active, in the case of a policy with a sufficiently large lag, an economy can be locally unstable.

2.2.4. ***Numerical example.*** Let us consider a numerical example. We set parameter values as $\varepsilon = 30$; $\delta = 15$; and $\tau = 0.5$. Further, we assume that the target inflation rate level and natural level of real national income are $\bar{\pi} = 0.02$ and $\bar{Y} = 100$, respectively. In addition, we specify the IS curve in (2.1) as $Y = 1/(R - \pi^e)$. Then, the steady-state values of R and π^e are given by $R^* = 0.03$ and $\pi^{e*} = 0.02$, respectively. Additionally, we obtain $Y'^* = -10000$.

Based on these specifications, we can derive Figure 1, which depicts the sets (θ_h, ω), where $h = 0, 1$, satisfying (2.17) on the θ-ω plane. These curves are referred to as the crossing curves.

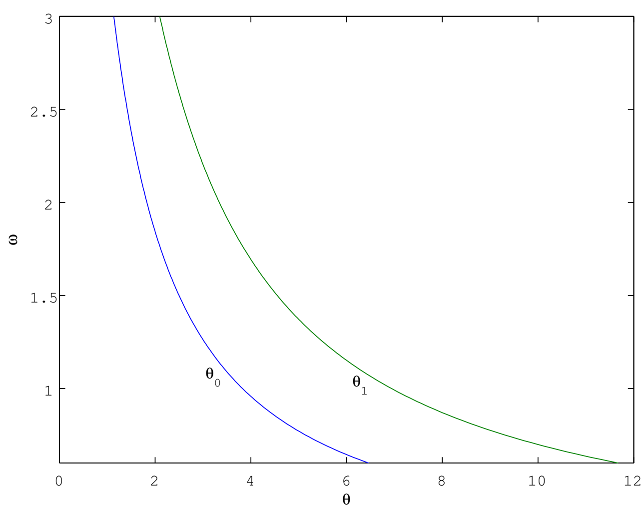

FIGURE 1. Crossing curves

In the region in the left-hand side of curve θ_0, a root with positive real parts does not exist, and accordingly, the steady state is stable. However, in the region on the right-hand side of curve θ_0, there exist at least two roots with positive real parts; therefore, the steady state is unstable. For example, when $\omega = 1.5$, the steady state destabilizes at $\theta = 2.499$.

3. New Keynesian Model

This section demonstrates that the aforementioned NK stability concept (*i.e.*, equilibrium determinacy) derives an atypical policy implication concerning economic stability, based on Tsuzuki(2014b, 2015). As is the case with the OK model, we examine two cases: the nonexistence and the existence of a policy lag.

3.1. Case of no policy lag. The NK model includes two types of approaches to money: the money in the utility function (MIUF) approach and the money in the production function (MIPF) approach. The economic implications of policy lags differ depending on which approach is adopted.[4]

In the standard NK model without a policy lag, a monetary policy rule is represented by $R = \bar{R} + s(\pi - \bar{\pi})$. The steady state becomes a saddle or a source, depending on the value of the political parameter, s, which represents the reactivity of the nominal interest rate to the inflation rate (See, *e.g.*, Asada(2013) and Galí(2015)):

(1) If $0 < s < 1$, the steady state is a saddle.
(2) If $1 < s$, the steady state is a source.

$0 < s < 1$ represents a passive monetary policy, whilst $1 < s$ represents an active one.

Both endogenous variables of this system (real national income and inflation rate) are jump variables. Therefore, according to the NK stability concept, in case 1, equilibrium is indeterminate. In contrast, in case 2, there exists a unique equilibrium path; in other words, there is no choice to set the initial values of the variables in the steady state. Hence, the equilibrium in this case is determinate. Accordingly, NK scholars consider case 2 as preferable to case 1.

In contrast, in the MIPF model, the steady state can become a saddle, a source, or a sink, depending on the values of the parameters. The dynamic property of the steady state of an MIPF model can be summarized as follows:

(1) If $0 < s < 1$, the steady state is a saddle.
(2) If $1 < s$ and
   (a) $s < \beta$, the steady state is a source,
   (b) $s > \beta$, the steady state is a sink,

where $\beta > 0$ is a value determined by structural parameters. According to the NK stability concept, in cases 1 and 2(b), the equilibrium is indeterminate, but in case 2(a), it is determinate.

Case of a positive policy lag. The policy rule is rewritten as: $R = \bar{R} + s(\pi(t - \vartheta) - \bar{\pi})$. We can develop an analysis similar to that performed for the OK model.

[4] According to Feenstra(1986) and Carldtrom and Fuerst(2003), an MIUF model with a negative correlation between consumption and real money balances (*i.e.*, $u_{cm} < 0$ when the utility function is expressed as $u(c, m)$) is equivalent to an MIPF model.

Under plausible parameter values, crossing curves as those shown in Figure 1 can be derived. (See Tsuzuki(2014b, 2015) for details.)

Particularly, in the case of a sink, an atypical policy implication can be drawn. For equilibrium determinacy, lag θ must take a positive value that is placed in the region between curves θ_0 and θ_1. This result suggests that the central bank should purposefully delay a policy implementation.

Notably, if we apply the traditional stability concept to the NK system, the implication of a policy lag for economic stability becomes the same as that of the OK model. In other words, if a lag exceeds a certain threshold level, an economy will become unstable. Thus, the difference in the results between OK and NK can be attributed to that of the stability concept.

4. Conclusions

This study examined the effect of a monetary policy lag on economic stability to explain the difference of policy implications between OK and NK models.

In the OK model with no policy lag, an economy is locally stable if the central bank is sufficiently active, *i.e.*, the reactivity of the nominal interest rate to the inflation rate is sufficiently large. However, in the case of a policy with a lag, if a lag exceeds a certain threshold level, an economy will become unstable, even though the central bank is sufficiently active.

On the other hand, in the NK model with the MIPF approach, if the central bank implements a policy without a lag and is active enough to satisfy the condition of $s < \beta$, then an equilibrium is determinate. However, in the case of a policy with a lag, equilibrium is determinate, if the lag is modest.

Despite the difference of their stability concepts, the OK and NK models derive the same policy implication in the case of no policy lag: the central bank should actively implement a monetary policy to stabilize an economy. However, in the case of a positive policy lag, the OK and NK models derive different implications. According to the OK model, the central bank should reduce a policy lag as much as possible. In contrast, according to the NK model, the central bank should seek a modest lag.

This difference in conclusion can be attributed to that between the stability concepts of OK and NK models. In fact, applying the traditional stability concept to the NK model, we can conclude that the central bank should reduce the policy lag.

Thus, our study demonstrates that considering a policy lag highlights the problem of the NK stability concept.

Appendix A. Appendix

A.1. Direction of crossing. Differentiating (2.12) with respect to θ, we obtain

$$\left(\frac{d\lambda}{d\theta}\right)^{-1} = -\frac{[2\lambda - \delta\{-\tau - (1-\tau)\varepsilon Y'^*\}]e^{\theta\lambda} - \omega\varepsilon Y'^*}{\omega\varepsilon Y'^*(\lambda+\delta)\lambda} - \frac{\theta}{\lambda},$$

where $e^{\theta\lambda}$ is derived from (2.12) as follows:

$$e^{\theta\lambda} = -\frac{\omega\varepsilon Y'^*(\lambda+\delta)}{-\lambda^2 + \delta\{-\tau - (1-\tau)\varepsilon Y'^*\}\lambda}.$$

Thus, we obtain

$$\begin{aligned}
\operatorname{sign}\left[\frac{d\mathrm{Re}\lambda}{d\theta}\right]_{\lambda=iz_+} &= \operatorname{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\theta}\right)^{-1}\right]_{\lambda=iz_+} \\
&= \operatorname{sign}\left[\mathrm{Re}\left\{\frac{2\lambda - \delta\{-\tau-(1-\tau)\varepsilon Y'^*\}}{\lambda(-\lambda^2 + \delta\{-\tau-(1-\tau)\varepsilon Y'^*\}\lambda)} + \frac{1}{\lambda(\lambda+\delta)} - \frac{\theta}{\lambda}\right\}\right]_{\lambda=iz_+} \\
&= \operatorname{sign}\left[\mathrm{Re}\left\{\frac{2z_+i - \delta\{-\tau-(1-\tau)\varepsilon Y'^*\}}{z_+i(z_+^2 + \delta\{-\tau-(1-\tau)\varepsilon Y'^*\}z_+i)} + \frac{1}{z_+i(z_+i+\delta)} - \frac{\theta_h}{iz_+}\right\}\right].
\end{aligned}$$

The last term of the right-hand side of the bottom equality, $-\theta_h/iz_+$, is obviously imaginary and therefore can be ignored.

Defining u and v as

$$u \equiv \frac{2z_+i - \delta\{-\tau-(1-\tau)\varepsilon Y'^*\}}{z_+i(z_+^2 + \delta\{-\tau-(1-\tau)\varepsilon Y'^*\}z_+i)}, \quad v \equiv \frac{1}{z_+i(z_+i+\delta)},$$

we can obtain

$$\mathrm{Re}\,u = \frac{\delta^2\{-\tau-(1-\tau)\varepsilon Y'^*\}^2 + 2z_+^2}{\delta^2\{-\tau-(1-\tau)\varepsilon Y'^*\}^2 z_+^2 + z_+^4}, \quad \mathrm{Re}\,v = -\frac{1}{z_+^2+\delta^2}.$$

Using these expressions, we obtain

$$\begin{aligned}
\operatorname{sign}\left[\frac{d\mathrm{Re}\lambda}{d\theta}\right]_{\lambda=iz_+} &= \operatorname{sign}\left[\mathrm{Re}\,u + \mathrm{Re}\,v\right], \\
&= \operatorname{sign}\left[\frac{z_+^4 + \delta^2[\delta^2\{-\tau-(1-\tau)\varepsilon Y'^*\}^2 + 2z_+^2]}{[\delta^2\{-\tau-(1-\tau)\varepsilon Y'^*\}^2 z_+^2 + z_+^4](z_+^2+\delta^2)}\right] > 0.
\end{aligned}$$

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