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A Sufficient Stability Condition for Fiscal Policy in a Business Cycle Model with Fuzziness

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ABSTRACT. This paper presents a macrodynamic fuzzy model with a fiscal stabilization policy. We evaluate the feasibility of asymptotic stabilization of production level through a feedback-type policy rule based on vague evaluation of the economic conditions. This vagueness is expressed as fuzzy inferences of whether an economy is in a boom or slump based on levels of production and capital stock. The fuzzy inferences are formalized via Takagi-Sugeno approach. Within our framework, the stability condition is investigated in order to clarify the feasibility of stabilization. Applying the Lyapunov theorem, we derive a sufficient condition for asymptotic stability.

***JEL Classification*:** C62, E12, E62

***Keywords*:** Fiscal policy; Stability condition; Lyapunov's direct method

1. Introduction

In *A Treatise on Probability*, Keynes argued that the concept of probability is defined as logical relations between two sets of propositions that constitute its inference, criticizing the frequency theory of probability. Within this perception, Keynes made a case that there is a relativity of probability concept due to limits of human reason. Keynes said:

> Probability is, *vide* Chapter II. (§12), relative in a sense to the principles of *human* reason. The degree of probability, which it is rational for *us* to entertain, does not presume perfect logical insight, and is relative in part to the secondary propositions which we in fact know; and it is not dependent upon whether more perfect logical insight is or is not conceivable. It is the degree of probability to which those logical processes lead, of which our minds are capable; or, in the language of Chapter II., which those secondary propositions justify, which we in fact know. If we do not take this view of probability, if we do not limit it in this way and make it, to this extent, relative to human powers, we are altogether adrift

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> in the unknown; for we cannot ever know what degree of probability would be justified by the perception of logical relations which we are, and must always be, incapable of comprehending. (Keynes,1921, pp.32-3.)

This assertion is exactly what Ramsey(1926) criticized. Indeed, Keynes failed to construct a convincing formalization of probability from the above standpoint and accepted Ramsey's criticism to some extent, but this fundamental thought was held after that. In chapter 12 of *General Theory*, for instance, Keynes made a distinction between "very uncertain" and "improbable" based on the idea of the weight of arguments developed in *A Treatise on Probability*. While Ramsey's thought on subjective probability has become a standard way of thinking in modern economics through Savage(1954)'s axiomatization, Keynes's philosophical position as a logical-relationist does not necessarily exclude the subjectivity in probability, as the above quotation implies.

From methodological angles, an affinity between fuzzy logic and Keynes's line of thought on probability has been pointed out by some researchers (see, *e.g.*, Coates (1997) and Dow(2009)). Fuzzy logic is a type of many-valued logic, which is developed with the intention of quantifying and reasoning natural languages. Therefore, it should be verified whether fuzzy logic is adaptable to post-Keynesian views. Meanwhile, a critique of the so-called Harvey Road presumption is well known. Since Keynes did not explicitly mention limits of recognition ability of policy makers, there is some validity in arguments in favor of automatic policy rules such as Phillips-type policy. On the other hand, some subjectivity or discretionarity is unavoidable in many cases of fiscal policy decision. Since it is very difficult to accomplish a formalization of economic behavior under radical uncertainty, it should also be investigated whether stabilization through fiscal policy is feasible even in situations where some subjectivity is involved in evaluation of the economic conditions.

Therefore, the main purposes of the present paper are to present a formalization of macrodynamics with fuzzy logic as an alternative for the logical-relationist theory and to verify the feasibility of fiscal stabilization policy when policy decisions are accompanied by vague judgments. As one of the few studies on fuzzy theory approach to policy theory, Keller(2009) has introduced fuzziness on policy implementation into continuous time models such as Phillips model and Goodwin model. However, the condition to ensure the stability has not been elucidated. We therefore develop a discrete time model of macrodynamics with fuzziness in economic evaluation of whether an economy is in a boom or slump. Unlike conventional optimization policy models, our fuzzy model does not require perfect knowledge of economic structure. Within our framework, the stability condition is investigated in order to verify the feasibility of stabilization through a feedback-type fiscal policy. We derive a sufficient condition for asymptotic stability by applying Lyapunov's direct method.

2. Fuzzy Sets and Generalized Modus Ponens

This chapter explains the concept of fuzzy sets and a generalized modus ponens. Fuzzy sets and fuzzy inference are generalization of the concepts of sets and modus ponens, respectively (see Zadeh(1965) and Zadeh(1973)). Conventionally, a characteristic function expresses whether an element belongs to the set or not. Let X be any classical set and let S be any subset of X. The characteristic function of S is as follows:

$$\chi_S(x) : X \to \{0, 1\}, \tag{2.1}$$

such that

$$\chi_S(x) = \begin{cases} 0, & x \notin S, \\ 1, & x \in S. \end{cases} \tag{2.2}$$

In fuzzy theory, this function is generalized to take values in the real closed interval [0, 1] in order to express various degrees of belonging, *i.e.*,

$$\mu_F(x) : X \to [1, 0]\,. \tag{2.3}$$

Such a function is called a membership function, by which a fuzzy set F is defined. By contrast, a set defined by a characteristic function is called a crisp set.

Let A, A^*, B and B^* be fuzzy sets. Suppose that A and B are similar to A^* and B^*, respectively. Modus ponens is generalized as follows (see, *e.g.*, Mendel(1995)):

$$\begin{aligned} \text{Premise 1:}&\ \ u \text{ is } A^*, \\ \text{Premise 2:}&\ \ \text{IF } u \text{ is } A \text{ THEN } v \text{ is } B, \\ \text{Consequence:}&\ \ v \text{ is } B^*, \end{aligned} \tag{2.4}$$

where the relation "is" indicates that an element belongs to a fuzzy set. Simple examples of fuzzy inference are as follows:

(2.5)

IF an economy is in a slump	THEN a high level of government spending is needed;
IF an economy is in a bad slump	THEN a very high level of government spending is needed;

where linguistic expressions, "slump" and "bad slump" mean that the economic conditions belong to fuzzy sets defined by membership functions representing subjective evaluation of economic slump. Similarly, "high" and "very high" express fuzzy sets on the level of government spending. In particular, Premise 2 is called the IF-THEN rule (see, *e.g.*, Dubois and Prade(1991)). The consequent parts of IF-THEN rules can be represented with functions (Takagi and Sugeno(1985)). In the next chapter, we put functions representing economic models into consequent parts. In addition, one can yield numerical results of fuzzy inferences by using defuzzification methods, as shown in the next chapter.

3. Model

3.1. Case with no fuzziness. This section presents a discrete time model of macrodynamics with a feedback-type fiscal policy. We focus on a restoring process of stationary equilibrium when there is a temporarily deviation of economic variables from equilibrium. Let economic variables be real-valued functions of a real variable t representing time period.

$Y(t)$ denotes the amount of production at time period t. The adjustment process in the goods market is defined by

$$Y(t+1) = Y(t) + \alpha(Y^D(t) - Y(t)), \tag{3.1}$$

where $Y^D(t)$ denotes the amount of demand at time period t, and α is the adjustment speed of disequilibrium ($0 < \alpha < 1$). In a manner of Phillips(1954), we interpret $Y(0)$ as a desired stationary equilibrium level of production, and hence $Y(0)$ is taken as a reference of the equilibrium, $Y(0) = 0$. Thus, $Y(t)$ represents the amount of deviation from equilibrium. $Y^D(t)$ is the sum of consumption $C(t)$, net investment $I(t)$, and government spending $G(t)$, *i.e.*,

$$Y^D(t) = C(t) + I(t) + G(t). \tag{3.2}$$

The amount of consumption is proportional to production level:

$$C(t) = (1-s)Y(t), \tag{3.3}$$

where s is a positive constant representing the saving rate ($0 < s < 1$).

$K(t)$ represents the amount of capital stock. We abstract depreciation of capital. The relation between $K(t)$ and $I(t)$ is defined by

$$K(t+1) = K(t) + I(t). \tag{3.4}$$

$K(0) = 0$ is a desired stationary equilibrium level of capital stock.

Let us introduce the investment function proposed by Kalecki(1935). For simplicity, we ignore a constant term of investment and suppose that there is no time lag between decision and spending on investment (see, *e.g.*, Allen(1966) and Lange(1970)). Then, Kalecki-type of investment function is given by

$$I(t) = \beta s Y(t) - \gamma K(t). \tag{3.5}$$

The positive coefficients β and γ express the sensitivity of investment decision to total savings and to capital stock, respectively. Let $O(t)$ and $Q(t)$ be the investment order and the profit, respectively. $Q(t)$ is defined by the sum of capitalists' consumption and accumulation. Kalecki assumes that the ratio of the amount of investment order to that of capital is an increasing function of the profit rate:

$$\frac{O(t)}{K(t)} = \phi\left(\frac{Q(t)}{K(t)}\right), \tag{3.6}$$

where ϕ denotes an linear increasing function. Hence, $O(t)$ can be written as

$$O(t) = \beta Q(t) - \gamma K(t). \tag{3.7}$$

Kalecki assumes that only capitalists make savings and decide the amount of investment. According to an interpretation of this function in Allen(1966) and Lange(1970), the investment order depends on non-capitalists' savings. Then, the investment order is described as

$$O(t) = \beta s Y(t) - \gamma K(t). \tag{3.8}$$

In addition, if there is no time lag between net investments and investment orders, [1] then one can put $O(t) = I(t)$ and obtain (3.5).

The parameter β is an important factor for asymptotic stability. Lange(1970) showed that in Kalecki's model the necessary condition for asymptotic stability is $\beta < 1$. This means that a sensitivity of investment to total savings must be less than 1, and hence a stable economy with no stabilization policy satisfies $I(t) < sY(t)$.

[1] In Kalecki(1935), the payment of investment are spread throughout the period of production of capital goods. This assumption gives the following expression:

$$I(t) = \frac{1}{\theta}\int_{t-\theta}^{t} O(\tau)d\tau,$$

where θ is a gestation period of investment.

In order to clarify the interpretation of our model, let us make a comparison between Lange's stability condition and the so-called Keynesian stability condition.[2] If the principle of effective demand is understood in the sense that investment is unconstrained by savings, then we should specify the investment function as

$$I(t) = \beta Y(t) - \gamma K(t), \tag{3.9}$$

instead of (3.5). Suppose that $G(t) = 0$ and $K(t) > 0$. Using the goods market clearing condition $Y(t) = C(t) + I(t)$ in (3.5), we obtain $I(t) = \frac{\gamma}{\beta - 1} K(t)$. Similarly, we obtain $I(t) = \frac{\gamma s}{\beta - s} K(t)$ from (3.9). Thus, in order to obtain positive values of $I(t)$, the conditions $1 < \beta$ and $s < \beta$ must be satisfied in (3.5) and (3.9), respectively. The former, $\beta < 1$, does not satisfy Lange's stability condition. The latter, $s < \beta$, is the instability condition which is always imposed in Kaldorian macrodynamic models; the condition that the saving rate is greater than the sensitivity of investment to production level, $\beta < s$, is often called the Keynesian stability condition. However, within our setting, negative values of $I(t)$ caused by the condition $\beta < 1$ are not necessarily unrealistic. This is because such negative values mean decreases in investment from equilibrium. Furthermore, note that our stabilization problem does not require Lange's stability condition, $\beta < 1$. This paper focuses on (3.5) for comparison with Phillips's disequilibrium model, but our discussion on stability can be applied to (3.9). It is easy to see that (3.5) corresponds to an extension of the accelerator-type investment of Phillips(1954). Phillips's model is a continuous-time model. If there is no gestation lag of investment, then the relation between investment and production level corresponds to that of a lagged accelerator. The accelerator relation presented by Phillips is written as follows:

$$I'(t) = \frac{1}{\tau_I} \left(kY'(t) - I(t) \right), \tag{3.10}$$

where τ_I and k represent the speed of accelerator lag and the acceleration coefficient, respectively. From (3.5) and $K'(t) = I(t)$, we obtain

$$\begin{aligned} I'(t) &= \beta s Y'(t) - \gamma K'(t) \\ &= \gamma \left(\frac{\beta s}{\gamma} Y'(t) - I(t) \right). \end{aligned} \tag{3.11}$$

The acceleration coefficient is $\beta s/\gamma$ and the coefficient of lag speed is $1/\gamma$ (see Allen (1966)).

[2] This issue was pointed out by an anonymous referee. The author is indebted to the referee's insightful comments.

Now, the target of policy is to achieve asymptotic stability of the equilibrium. The government spending is proportionally adjusted in order to asymptotically stabilize the time path of the economic variables. $G(t)$ is defined by the following feedback policy based on deviation from the initial equilibrium value $Y(0)$:

$$G(t) = -g\,[Y(t) - Y(0)] = -gY(t), \tag{3.12}$$

where g is a positive constant. $Y(t)$ and $K(t)$ are summarized in the following matrix form:

$$E(t+1) = AE(t) + BG(t), \tag{3.13}$$

where $E(t)^T = (Y(t), K(t))$, $A \in R^{2\times 2}$, $B \in R^{2\times 1}$. R denotes the set of real numbers. The symbol ^{T} indicates transposition of matrices. From the above relations, we have

$$\begin{pmatrix} Y(t+1) \\ K(t+1) \end{pmatrix} = \begin{pmatrix} 1 + \alpha s(\beta - 1) & -\alpha\gamma \\ \beta s & 1 - \gamma \end{pmatrix} \begin{pmatrix} Y(t) \\ K(t) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} G(t). \tag{3.14}$$

It is easy to check the controllability by seeing that the rank of a matrix $[B \mid BA]$ is of full row rank. Similarly, (3.12) is restated as follows:

$$G(t) = PE(t), \tag{3.15}$$

where $P = (-g, 0) \in R^{1\times 2}$. The equilibrium point is trivially given by the origin, and we investigate the global asymptotic stability of the origin.

3.2. Fuzzy model. We consider a situation that one can obtain only linear approximation models with some subjectivity because the real economy has a complex behavior with nonlinearity. Let us formalize fuzziness in evaluation of the economic conditions in order to take such subjectivity or discretionarity into account. We consider the formalization developed in previous section to be a local linear approximation of nonlinear behavior. Specifically, we suppose that the levels of investment and consumption depend on the economic conditions. In such a situation, policy maker may presume that the values of structural parameters β, γ, s depend on the levels of production and capital stock. We assume that the subjective evaluation of whether an economy is in a boom or slump is based on certain high and low levels of $Y(t)$ and $K(t)$. Keynes of course characterized the boom and the slump more precisely (see, *e.g.*, Ch. 22 in *General theory*), but it is important to investigate the feasibility of stabilization policy based on a simpler evaluation of economy in order to demonstrate the significance of policy implementation itself. Suppose that $\overline{Y}$ and $\overline{K}$ represent such reference levels of economic boom on production and capital stock, respectively. Similarly, $\underline{Y}$ and

$\underline{K}$ represent reference levels of economic slump. We assume that $\underline{Y} < 0 < \overline{Y}$ and $\underline{K} < 0 < \overline{K}$.

Now, we consider four fuzzy sets on the economic conditions defined by each membership function. Suppose that fuzzy sets H_1, L_1, H_2, and L_2 express the subjective degrees of "the value of $Y(t)$ is high," "the value of $Y(t)$ is low," "the value of $K(t)$ is high," "the value of $K(t)$ is low," respectively. For example, we can describe their membership functions as follows:[3]

$$\mu_{H_1}(Y(t)) = \begin{cases} 0, & Y(t) < \underline{Y}, \\ \dfrac{Y(t) - \underline{Y}}{\overline{Y} - \underline{Y}}, & \underline{Y} \leq Y(t) \leq \overline{Y}, \\ 1, & Y(t) > \overline{Y}, \end{cases} \tag{3.16}$$

$$\mu_{L_1}(Y(t)) = \begin{cases} 0, & Y(t) > \overline{Y}, \\ \dfrac{\overline{Y} - Y(t)}{\overline{Y} - \underline{Y}}, & \underline{Y} \leq Y(t) \leq \overline{Y}, \\ 1, & Y(t) < \underline{Y}, \end{cases} \tag{3.17}$$

$$\mu_{H_2}(K(t)) = \begin{cases} 0, & K(t) < \underline{K}, \\ \dfrac{K(t) - \underline{K}}{\overline{K} - \underline{K}}, & \underline{K} \leq K(t) \leq \overline{K}, \\ 1, & K(t) > \overline{K}, \end{cases} \tag{3.18}$$

$$\mu_{L_2}(K(t)) = \begin{cases} 0, & K(t) > \overline{K}, \\ \dfrac{\overline{K} - K(t)}{\overline{K} - \underline{K}}, & \underline{K} \leq K(t) \leq \overline{K}, \\ 1, & K(t) < \underline{K}. \end{cases} \tag{3.19}$$

From (3.13), IF-THEN rules for modeling of the economic structure are written as follows:

$$\begin{aligned} &Rule\,1: \text{ IF } Y(t) \text{ is } H_1 \text{ and } K(t) \text{ is } H_2 \text{ THEN } E_1(t+1) = A_1E(t) + BG(t), \\ &Rule\,2: \text{ IF } Y(t) \text{ is } H_1 \text{ and } K(t) \text{ is } L_2 \text{ THEN } E_2(t+1) = A_2E(t) + BG(t), \\ &Rule\,3: \text{ IF } Y(t) \text{ is } L_1 \text{ and } K(t) \text{ is } H_2 \text{ THEN } E_3(t+1) = A_3E(t) + BG(t), \\ &Rule\,4: \text{ IF } Y(t) \text{ is } L_1 \text{ and } K(t) \text{ is } L_2 \text{ THEN } E_4(t+1) = A_4E(t) + BG(t). \end{aligned} \tag{3.20}$$

[3] Here, we defined $w_i(t)$ in such a way that for all i, $w_i(t) \geq 0$, and $\sum_{i=1}^{4} w_i(t) > 0$.

“$Y(t)$ is H_1” and “$Y(t)$ is L_1” mean that the value of $Y(t)$ belongs to a fuzzy set H_1 (*i.e.*, the value of $Y(t)$ is high) and L_1 (*i.e.*, the value of $Y(t)$ is low), respectively. Similarly, “$K(t)$ is H_2” and “$K(t)$ is L_2” mean that the value of $K(t)$ belongs to a fuzzy set H_2 (*i.e.*, the value of $K(t)$ is high) and L_2 (*i.e.*, the value of $K(t)$ is low), respectively.

As mentioned above, the parameters β, γ and s depend on the levels of production and capital stock. Then, the matrices, A_1, A_2, A_3 and A_4, are described as

$$A_1 = \begin{pmatrix} 1+\alpha s_1(\beta_1-1) & -\alpha\gamma_1 \\ \beta_1 s_1 & 1-\gamma_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1+\alpha s_2(\beta_2-1) & -\alpha\gamma_2 \\ \beta_2 s_2 & 1-\gamma_2 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 1+\alpha s_3(\beta_3-1) & -\alpha\gamma_3 \\ \beta_3 s_3 & 1-\gamma_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1+\alpha s_4(\beta_4-1) & -\alpha\gamma_4 \\ \beta_4 s_4 & 1-\gamma_4 \end{pmatrix}.$$

The weight implying the overall truth value of the each premise in (3.20) is given by the product of values of membership functions as follows:

$$w_1(t) = \mu_{H_1}(Y(t))\mu_{H_2}(K(t)), \tag{3.21}$$
$$w_2(t) = \mu_{H_1}(Y(t))\mu_{L_2}(K(t)), \tag{3.22}$$
$$w_3(t) = \mu_{L_1}(Y(t))\mu_{H_2}(K(t)), \tag{3.23}$$
$$w_4(t) = \mu_{L_1}(Y(t))\mu_{L_2}(K(t)). \tag{3.24}$$

In order to yield numerical results of fuzzy inferences, various defuzzification methods are proposed. In particular, the center of gravity method is well known (see, *e.g.*, Hong(1996)). On the basis of this method, we obtain a economic model as a consequence of fuzzy inferences with weights $w_i(t)$ as follows:

$$E(t+1) = \frac{\sum_{i=1}^{4} w_i(t)(A_i E(t) + BG(t))}{\sum_{i=1}^{4} w_i(t)}. \tag{3.25}$$

Then, from (3.15) we have

$$E(t+1) = \frac{\sum_{i=1}^{4} w_i(t)(A_i + BP)E(t)}{\sum_{i=1}^{4} w_i(t)}. \tag{3.26}$$

Furthermore, if needed, the model can be extended to include some discretionarity as fuzziness in adjustments of policy rule. In such a case, in addition to fuzziness in approximation of economic structure, we need to formalize fuzziness in economic evaluation in which a policy rule is adjusted. For example, when policy maker adjusts the policy rule in four ways based on vague judgments of whether the economy is in a boom or slump, we can construct a fuzzy expression of government spending rules of the form

$$G(t) = \frac{\sum_{j=1}^{4} \hat{w}_j(t) P_j E(t)}{\sum_{j=1}^{4} \hat{w}_j(t)}, \tag{3.27}$$

where $\hat{w}_j(t)$ represents the weight in rule j. The weight $\hat{w}_j(t)$ does not have to be the same as $w_i(t)$. Substituting this into (3.25), we obtain a model with fuzziness on policy adjustment as follows:

$$E(t+1) = \frac{\sum_{i=1}^{4} \sum_{j=1}^{4} w_i(t)\hat{w}_j(t)(A_i + BP_j)E(t)}{\sum_{i=1}^{4} \sum_{j=1}^{4} w_i(t)\hat{w}_j(t)}. \tag{3.28}$$

The next chapter will focus on the stability of model (3.26), but a similar argument can be made for the stability of this extended model.

4. Stability Condition

In this section, we derive a sufficient condition to ensure the asymptotic stability of the fuzzy economic model (3.26) by applying Lyapunov's direct method (see, *e.g.*, Tanaka and Sugeno(1992)). Let '$C > \mathbf{0}$' denote that C is a positive definite matrix.

PROPOSITION 4.1. *If there exists a common positive definite matrix* C *such that*

$$(A_i + BP)^T C(A_i + BP) - C < \mathbf{0}, \; i = 1, 2, 3, 4, \tag{4.1}$$

then the equilibrium of a fuzzy economic model (3.26) *is globally asymptotically stable.*

Proof. Let $C \in R^{2\times 2}$ be a positive definite matrix. First, we see that if $(A_i + BP)^T C(A_i + BP) - C < \mathbf{0}$ and $(A_j + BP)^T C(A_j + BP) - C < \mathbf{0}$, then

$$(A_i + BP)^T C(A_j + BP) + (A_j + BP)^T C(A_i + BP) - 2C < \mathbf{0}. \tag{4.2}$$

We have

$$\begin{aligned}
&(A_i + BP)^T C(A_j + BP) + (A_j + BP)^T C(A_i + BP) - 2C \\
&= -\{(A_i + BP) - (A_j + BP)\}^T C\{(A_i + BP) - (A_j + BP)\} \\
&+(A_i + BP)^T C(A_i + BP) + (A_j + BP)^T C(A_j + BP) - 2C.
\end{aligned} \tag{4.3}$$

Since $C > \mathbf{0}$, it follows that

$$-\{(A_i + BP) - (A_j + BP)\}^T C\{(A_i + BP) - (A_j + BP)\} \leq \mathbf{0}. \tag{4.4}$$

Hence, we obtain (4.2).

Let us see that a continuous scalar function, $V(E(t)) = E^T(t)CE(t)$, is a Lyapunov function. It is easy to see that

$$\begin{aligned} &V(\vec{0}) = 0, \\ &V(E(t)) > 0 \text{ for } E(t) \neq \vec{0}, \\ &V(E(t)) \to \infty \text{ as } \|E(t)\| \to \infty, \end{aligned} \tag{4.5}$$

where $\vec{0}$ denotes a zero vector. From (3.26), we obtain

$$\begin{aligned} V(E(t+1)) - V(E(t)) &= E^T(t+1)CE(t+1) - E^T(t)CE(t) \\ &= \left(\frac{\sum_{i=1}^4 w_i(t)(A_i + BP)E(t)}{\sum_{i=1}^4 w_i(t)}\right)^T C \left(\frac{\sum_{i=1}^4 w_i(t)(A_i + BP)E(t)}{\sum_{i=1}^4 w_i(t)}\right) \\ &\qquad - E^T(t)CE(t) \\ &= \frac{1}{W}\Big[\sum_{i=1}^4 w_i^2(t)E^T(t)\{(A_i + BP)^T C(A_i + BP) - C\}E(t) \\ &\quad + \sum_{i=1}^4 \sum_{j=1}^4 w_i(t)w_j(t)E^T(t)\{(A_i + BP)^T C(A_j + BP) \\ &\qquad + (A_j + BP)^T C(A_i + BP) - 2C\}E(t)\Big], \end{aligned} \tag{4.6}$$

where $W = \sum_{i=1}^4 \sum_{j=1}^4 w_i(t)w_j(t)$. It follows from (4.1) and (4.2) that $V(E(t+1)) - V(E(t)) < 0$, and hence $V(E(t))$ is a Lyapunov function. Thus, from the Lyapunov theorem, a fuzzy economic model (3.26) is globally asymptotically stable if the condition (4.1) is satisfied. □

This theorem shows that the economy (3.26) is not generally stable even if all the linear subsystems $E_i(t+1) = (A_i + BP)E(t)$ are stable. If each linear subsystem can be stabilized through fiscal policy under $m_i < g < n_i$, then we must choose a policy parameter such that $\max_i m_i < g < \min_i n_i$ in order to stabilize the economy (3.26) at least. Instead, as long as there exists a common positive definite matrix C, the economy (3.26) is stable regardless of the shape of membership functions. This means that the existence of discretionarity as fuzziness in economic evaluation does not necessarily lead to impossibility of stabilization. Under this sufficient condition for asymptotic stability, an instable economy is stabilized through fiscal policy, or in a stable economy the time path of the economy can be improved in view of the fluctuation and the speed of convergence to equilibrium.

We now give a numerical example. For simplicity, we investigate a case where a fuzzy model consists of two fuzzy rules. Suppose that $g = 0.5$, $\alpha = 0.5$, $s_1 = 0.2$, $\beta_1 = 1.4$, $\gamma_1 = 0.1$, $s_2 = 0.4$, $\beta_2 = 0.7$, $\gamma_2 = 0.2$. Then, we have

$$A_1 + BP = \begin{pmatrix} 0.79 & -0.05 \\ 0.28 & 0.90 \end{pmatrix}, \quad A_2 + BP = \begin{pmatrix} 0.69 & -0.10 \\ 0.28 & 0.80 \end{pmatrix}. \tag{4.7}$$

We choose

$$C = \begin{pmatrix} 2.85 & 0.77 \\ 0.77 & 2.51 \end{pmatrix} \tag{4.8}$$

such that

$$(A_2 + BP)^T C(A_2 + BP) - C = -\mathbf{1}, \tag{4.9}$$

where $\mathbf{1}$ denotes the identity matrix. Clearly, we have $(A_2+BP)^T C(A_2+BP)-C < \mathbf{0}$. In addition, since the eigenvalues of $(A_1 + BP)^T C(A_1 + BP) - C$ are -0.83 and -0.25, we obtain

$$(A_1 + BP)^T C(A_1 + BP) - C = \begin{pmatrix} -0.53 & 0.29 \\ 0.29 & -0.54 \end{pmatrix} < \mathbf{0}. \tag{4.10}$$

Hence, this economy is stable: the fiscal policy with $g = 0.5$ is a stable intervention for this economy.

5. Conclusions

We have constructed a macrodynamic model with a feedback-type fiscal policy based on vague evaluation of the economic conditions, as an alternative of Keynes's line of thought on probability and discretionary policy. The vagueness is expressed as fuzzy inferences of whether an economy is in a boom or slump based on levels of production and capital stock via Takagi-Sugeno approach. On the basis of the Lyapunov theorem, we have clarified the feasibility of stabilization through a feedback-type fiscal policy. The existence of discretionarity as fuzziness in economic evaluation does not necessarily lead to impossibility of stabilization. The time path of the economy can be asymptotically stabilized through fiscal policy as long as the stability condition is satisfied. A numerical example has shown that a stable intervention is feasible under the existence of fuzziness. Our model does not treat the government budget constraint. Although it is beyond the scope of this paper, it should be noted that our formalization provides a basis for description of a case where the government spending is financed by tax revenues, government bonds, and/or government notes.

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