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A Mathematical Note on Equilibrium Concepts of Neoclassical and Post-Keynesian Economics

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ABSTRACT. This article examines the concept of equilibrium and stability in macrodynamics. We compare Post-Keynesians and Neoclassicals. Our major points are three: First, Post-Keynesian's concept of stability of equilibrium is formally natural one. Second, neoclassical saddle point equilibrium with indeterminacy may suffer from fatal difficulties. Third, some Ramsey-type models with externalities can have equilibrium other than the saddle point, which suggests the relationship between saddle point equilibrium and perfect market mechanism.

The main purposes of this study are as follows. Models based on intertemporal optimization by a representative household generate dynamics that are hard to interpret and justify: (i) saddle point soultions may have determinate outcomes, but it is unreasonable to assume that agents are able to calculate and jump straight to the stable saddle path, and (ii) stable solutions may arise in some cases; stable solutions imply indeterminacy which means that there is no mechanism by which agents can coordinate on a rational expectations/perfect foresight trajectory. However, Keynesian models do not suffer from these weaknesses and the Keynesian models can generate a range of dynamic outcomes.

Keywords: Ramsey model, Kaldor model, saddle point, indeterminacy, externality, public goods

JEL Classification: E12; E32; E62

1. Introduction

The neoclassical theory of economic growth is based on the intertemporal optimization problem initiated by Ramsey(1928). Households or the social planner (government) must solve the problem so as to maximize the present value of discounted future utility levels under the market equilibrium. Neoclassical economists seem to like depicting optimistic future of market economies. Most of them discuss the existence of the saddle point path of their dynamic models, and stick themselves to the idea that the economy will converge to the saddle point equilibrium. This is

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against textbook mathematics, for the saddle point is unstable. Then, we have to ask: what is meant by the saddle point equilibrium of neoclassical models, and its implications to the market economy.

The plan of the article is as follows. Section 2 is dedicated to a Post-Keynesian model by Kaldor as a benchmark of our analysis. This is based on the equilibrium with mathematically natural sense of stability. Section 3 takes up the simplest and typical neoclassical model. We explain how neoclassical models claim that the saddle point will be reachable by introducing the concept of indeterminacy. Section 4 presents two Ramsey-type models with externalities, that have non-saddle equilibria. Section 5 presents our summary.

2. Post-Keynesian Macrodynamic Model

In this section, we examine the Kaldorian cycle model (Kaldor,1940).

Kaldor model. Disequilibrium of the goods market is allowed in the Kaldor model. We assume that the disequilibrium is adjusted only by variations in quantities of production; the adjustment by variations in prices is omitted. We examine the simplest case in which the aggregate demand consists of only consumption C_t and investment I_t. Then, the adjustment function of the goods market is represented by

$$Y_{t+1} - Y_t = \alpha(C_t + I_t - Y_t), \tag{2.1}$$

where Y_t stands for national income (output), and α $(0 < \alpha < 1)$ denotes the adjustment speed of disequilibrium. Y_t and C_t are net of depreciation.

If Y_t is smaller than the full-employment level Y_f, which is a given constant, then unemployment exists. Thus, this model allows for the existence of the involuntary unemployment.

We assume that household consumption depends on income:

$$C_t = cY_t + C_0, \tag{2.2}$$

where c $(0 < c < 1)$ denotes the marginal propensity to consume, and C_0 (> 0) stands for base consumption.

The volume of investment is independent of households savings, unlike in the neoclassical model, and we assume that the investment function is given by

$$I_t = I(Y_t, K_t), \tag{2.3}$$

where $\frac{\partial I_t}{\partial Y_t} > 0$ and $\frac{\partial I_t}{\partial K_t} < 0$. For simplicity, we specify (2.3) as

$$I_t = -Y_t^{-\beta} K_t^{\gamma} + I_0, \tag{2.4}$$

where I_0 (> 0) is an autonomous investment. We do not assume the sigmate investment function, but a simple one (2.4). It seems that the sigmate investment function was necessary for Kaldor(1940) to demonstrate the existence of periodic solutions.[1] Note also that the linear homogeneity of the investment function with respect to (Y, K) implies that the time scale is the middle run. The linear homogeneity, however, is avoidable by adding, if necessary, a constant to the investment function. This alteration makes no difference to our argument. It may also be worth noting that in the continuous-time version of Kaldor model, cycles are excluded if both investment and saving are linearly homogeneous in (Y, K) (Skott, 1989).

Assuming that no capital depreciation exists,[2] we obtain

$$K_{t+1} = K_t + I_t. \tag{2.5}$$

System and linearization. The system of (2.1), (2.2), (2.4) and (2.5) is reduced to the following system of difference equations:

$$\begin{aligned} Y_{t+1} &= (1-\alpha s)Y_t - \alpha Y_t^{-\beta} K_t^{\gamma} + \alpha I_0 + \alpha C_0, \\ K_{t+1} &= K_t - Y_t^{-\beta} K_t^{\gamma} + I_0, \end{aligned}$$

where $s = 1 - c$ stands for the marginal propensity to save.

The non-trivial steady states of Y_t and K_t are given by

$$Y^* = \frac{C_0}{s}, K^* = \left(I_0 Y^{*\beta}\right)^{\frac{1}{\gamma}}.$$

By linearizing the system in the vicinity of the steady state, we obtain

$$\begin{bmatrix} \hat{Y}_{t+1} \\ \hat{K}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \alpha s + \frac{\alpha\beta I_0}{Y^*} & -\frac{\alpha\gamma I_0}{Y^*} \\ \frac{\beta I_0}{K^*} & 1 - \frac{\gamma I_0}{K^*} \end{bmatrix} \begin{bmatrix} \hat{Y}_t \\ \hat{K}_t \end{bmatrix}, \tag{2.6}$$

where the variables marked by '^' represent the ratios of deviation from the steady state values, *e.g.*, $\hat{Y}_t = \frac{Y_t - Y^*}{Y^*}$.

Dynamic property of (2.6). The characteristic equation of (2.6), $F(x) = 0$, is given by

$$F(x) = x^2 + a_1 x + a_2 = 0, \tag{2.7}$$

where

$$a_1 = -\left(2 - \alpha s + \frac{\alpha\beta I_0}{Y^*} - \frac{\gamma I_0}{K^*}\right),\ a_2 = (1-\alpha s)\left(1 - \frac{\gamma I_0}{K^*}\right) + \frac{\alpha\beta I_0}{Y^*}.$$

[1] Chang and Smyth(1971) applied the Poincaré-Bendixson theorem in their proof. Asada(1997, Ch.3) showed that, without a sigmate investment function, periodic solutions exist for certain values of parameters, by applying the Hopf bifurcation theorem.

[2] No capital depreciation implies that a negative I_t in (2.4) means an obsolescence.

For the steady state to be locally stable, the absolute values of all roots of (2.7) must be less than 1 in view of the Schur-Cohn theorem, that is, the following A_js ($j = 1, 2, 3$) must be positive (Gandolfo,1996, Ch.5 and Asada, 2008, Ch.3).

$$A_1 = 1 + a_1 + a_2 = \alpha s \frac{\gamma I_0}{K^*},$$
$$A_2 = 1 - a_2 = \alpha s - \frac{\alpha\beta I_0}{Y^*} + (1 - \alpha s)\frac{\gamma I_0}{K^*},$$
$$A_3 = 1 - a_1 + a_2 = 4 + \alpha s \frac{\gamma I_0}{K^*}.$$

It is clear that $A_1 > 0$ and $A_3 > 0$. Hence, if $A_2 > 0$, the steady state is a sink; if $A_2 < 0$, then the steady state is a source.[3]

In terms of the elasticity β of investment with respect to income, the above can be read as follows. If β is sufficiently small, $A_2 > 0$ holds and the steady state is stable. However, if β is sufficiently large, $A_2 < 0$ holds and the steady state is unstable. These results can be summarized as follows:

PROPOSITION 2.1 (Kaldor stability). *The steady state of the Kaldor model is a sink when β is relatively small; it is a source when β is relatively large.*

Focusing on α, we may say the following: if α is sufficiently large ($\alpha > s + \frac{\beta K^*}{\gamma} - \frac{sK^*}{\gamma I_0}$), then $A_2 > 0$ and the steady state is stable even if $\beta = 0$.

Asada(1997) develops a continuous-time Kaldor model, which comprises a system of differential equations, and obtains the similar result.

3. Neoclassical Macrodynamic Model

In this section, we examine the simplest, but typical, neoclassical macrodynamic model with exogenous labor supply from the angle of dynamics.

3.1. Saddle point equilibrium. The model contains two types of economic agents: households and firms. Households consume goods and supply the firms with capital services and labor forces. Firms produce goods using capital and labor forces under perfect competition. We specify behaviors of the agents.

Households. Households obtain utility from consumption C_t at every period. We represent the utility function of households as

$$u(C_t) = \frac{C_t^{1-\sigma} - 1}{1 - \sigma}, \tag{3.1}$$

where σ (> 0) denotes the elasticity of the marginal utility of consumption.

[3] Since $F(-1) > 0$ and $F(1) > 0$, the steady-state point can never be a saddle point.

At the beginning of period t, households possess capital stock K_t. This capital stock K_t yields profit within a single period at a rate r_t. Households also supply labor forces L_t to firms and receive real wages w_t. Denoting savings as S_t, the inter-temporal budget constraint of households is written as

$$S_t = r_t K_t + w_t L_t - C_t. \tag{3.2}$$

Capital stock K_t is increased by investment I_t. For the sake of simplicity, we assume no capital wastage. Then, we obtain

$$K_{t+1} = K_t + I_t. \tag{3.3}$$

In addition, the equilibrium condition of the goods market is given by

$$I_t = S_t. \tag{3.4}$$

Combining (3.2), (3.3), and (3.4), we obtain

$$K_{t+1} = (1 + r_t)K_t + w_t L_t - C_t. \tag{3.5}$$

Given the restriction (3.5), households determine C_t to maximize the sum of the discounted present value of the stream of utility, which is given by

$$U_t = E_t \left[\sum_{t=0}^{\infty} \zeta^t \frac{C_t^{1-\sigma} - 1}{1 - \sigma} \right],$$

where ζ $(0 < \zeta < 1)$ is the subjective discount factor of households.

We assume that the volume of labor supply is constant, $L_t = L$, and that the initial value of capital stock K_o is a given constant.

The first-order condition for optimality is given by the Euler equation:

$$\frac{1}{\zeta} \left(\frac{C_{t+1}}{C_t} \right)^{\sigma} = 1 + r_{t+1}. \tag{3.6}$$

Firms. Firms produce goods Y_t by using capital stock K_t and labor forces L_t. The production function of firms is represented as

$$Y_t = K_t^{\nu} L_t^{1-\nu}, \tag{3.7}$$

where $0 < \nu < 1$.

Under perfect competition, firms maximize the profits, so that

$$r_t = \nu K_t^{\nu-1} L_t^{1-\nu}, \tag{3.8}$$

$$w_t = (1 - \nu) K_t^{\nu} L_t^{-\nu}. \tag{3.9}$$

System and linearization. (3.5), (3.6), (3.8), and (3.9) are reduced to the following system of difference equations:

$$\frac{1}{\zeta}\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = 1 + \nu K_{t+1}^{\nu-1} L^{1-\nu}, \tag{3.10}$$

$$K_{t+1} = K_t + K_t^{\nu} L^{1-\nu} - C_t. \tag{3.11}$$

The non-trivial steady-state values of C_t and K_t are given by

$$C^* = \left\{\frac{1}{\nu}\left(\frac{1}{\zeta}-1\right)\right\}^{\frac{\nu}{\nu-1}} L, \quad K^* = \left\{\frac{1}{\nu}\left(\frac{1}{\zeta}-1\right)\right\}^{\frac{1}{\nu-1}} L.$$

By linearizing the system in the vicinity of the steady state, we obtain

$$\begin{bmatrix}\hat{C}_{t+1}\\ \hat{K}_{t+1}\end{bmatrix} = \begin{bmatrix}1-\frac{\zeta(\nu-1)}{\sigma}(\frac{1}{\zeta}-1)^2\frac{1}{\nu} & \frac{\nu-1}{\sigma}(\frac{1}{\zeta}-1)\\ -\frac{1}{\nu}(\frac{1}{\zeta}-1) & \frac{1}{\zeta}\end{bmatrix}\begin{bmatrix}\hat{C}_t\\ \hat{K}_t\end{bmatrix}. \tag{3.12}$$

Dynamic property of (3.12). The characteristic equation of (3.12) is given by

$$F(x) = x^2 + b_1 x + b_2 = 0 \tag{3.13}$$

where

$$b_1 = -\left\{1 - \frac{\zeta(\nu-1)}{\sigma}\left(\frac{1}{\zeta}-1\right)^2\frac{1}{\nu} + \frac{1}{\zeta}\right\},\ b_2 = \frac{1}{\zeta}.$$

Since $B_1 = 1 + b_1 + b_2 = \frac{\zeta(\nu-1)}{\nu\sigma}(\frac{1}{\zeta}-1)^2 < 0$, one of the conditions of the Schur-Cohn theorem is not satisfied, and hence, the steady state is not a sink; in view of $F(-1) > 0$ and $F(1) < 0$, it is a saddle point,

We then establish the following proposition.

PROPOSITION 3.1 (Neoclassical saddle point equilibrium). *The steady state of a neoclassical perfect competition model characterized by* (3.1) *and* (3.7) *is a saddle point.*

3.2. Fatal difficulties of reachable saddle points. When a system is linear, the saddle path can be expressed by an explicit function, which is called a policy function. A policy function gives the values of the control variable ($\hat{C}_t$) as a function of the state variable ($\hat{K}_t$). Let the policy function of (3.12) be expressed as

$$\hat{C}_t = C(\hat{K}_t), \tag{3.14}$$

where the transversality condition, $\lim_{t\to\infty} \zeta^t K_t = 0$ is required (Barro and Sala-i-Martin, 2003). Since the saddle path becomes a line when a system is linear, (3.14) can be rewritten as $\hat{C}_t = \delta\hat{K}_t$, where δ is a certain constant.

Given an initial value of $\hat{K}_t$, denoted by $\hat{K}_o$, the initial value of $\hat{C}_t$, denoted by $\hat{C}_o$, is uniquely determined by the policy function. Once a set of initial values

$(\hat{K}_o, \hat{C}_o)$ is determined, an economy will converge to the steady state along the line with the saddle path (see Fig. 1).[4]

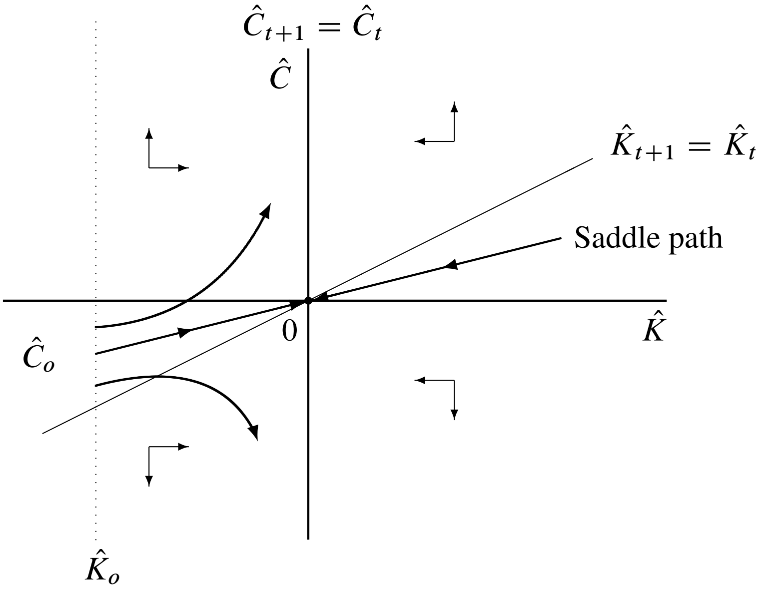

FIGURE 1. Saddle path

The first concerns the complexity of a policy function. If the system is nonlinear, as is actually so, however, and if we introduce multiple types of capital stock, such as human capital, it would become more difficult to obtain a policy function.

The second concerns the realism of the model. In the above households determine the path of C_t. However, it seems quite difficult for households to place C_o exactly on a saddle path. Fujimori(2009) pointed out a precision difficulty of the saddle path. These are just a common sense in textbooks of mathematics.

Then, why do the Neoclassicals regard a saddle point as stable? They claim as follows.

C_{t+1} in (3.10) is the mathematical expectation of consumption at period $t + 1$ evaluated at period t. Assuming the rational expectation, a policy function can be obtained. Thus, a saddle path is determined from the behavior of the forward-looking agents; namely, the future (C_{t+1}) determines the present (C_t). The initial value of consumption C_o is not a given constant but a value that must be determined within the model. If the steady state is a saddle point, C_o can be uniquely determined by the policy function for a given K_o. However, if the steady state is a sink, candidates for the initial value of C_o innumerably exist and cannot be determined uniquely. For this reason, the Neoclassicals bring up the idea of "indeterminacy."

[4] For better visuality, the continuous lines are drawn on behalf of dotted line.

The idea of indeterminacy has significant difficulties, as criticized by Asada(2013).

3.3. Critique of jump variables. There are two endogenous variables in the dynamic system of the neoclassical model presented in this paper: consumption C and capital stock K. Generally, C is regarded as a jump variable or a non-predetermined variable and K as a predetermined variable.

The jump variable is defined as a variable of which the initial value can be arbitrarily chosen by agents (in our model, households). On the other hand, the predetermined variable is defined as a variable of which the initial value is given and constant.

As stated above, the assumption that the initial value of a jump variable is exactly chosen on a path converging to a steady state (*i.e.*, saddle path) is extremely unrealistic. That assumption deserves criticism for *ad hoc*, as the Neoclassicals made sharp criticisms of the old Keynesians.

Asada(2013) makes a similar criticism of the stability concept of the Neoclassicals with a more mathematically precise argument, although his focus is on the so-called new Keynesian model that incorporates the stickiness of prices to the neoclassical model.

Quoting a few sentences from Mankiw(2001), Asada(2013) points out that the actual inflation rate does not discontinuously fluctuate, and hence, π should not be treated as a jump variable.

4. Non-saddle Point Equilibrium

In this section, we take up two neoclassical macro dynamic models, which have no saddle points, nonetheless.

4.1. Endogenous labor supply and externalities. Benhabib and Farmer(1994) introduce the assumptions of endogenous labor supply (*i.e.*, household choice between leisure and labor) and externalities of capital and labor. The utility function of households is represented as

$$u(C_t, L_t) = \ln C_t - \frac{L_t^{1+\chi}}{1+\chi}, \tag{4.1}$$

where χ (> 0) denotes the elasticity of the marginal disutility of labor. The term $-\dfrac{L_t^{1+\chi}}{1+\chi}$ represents the disutility caused by labor supply.

The production function of firms is represented as

$$Y_t = K_t^{\nu} L_t^{1-\nu} \bar{K}_t^{\varepsilon} \bar{L}_t^{\eta}, \tag{4.2}$$

where $\bar{K}_t$ is the externality of capital stock, $\bar{L}_t$ is the externality of labor forces, and ε and η represent the scale of the externalities of $\bar{K}_t$ and $\bar{L}_t$, respectively.

Under these assumptions, Benhabib and Farmer(1994) demonstrate that for certain values of ε and η, the Jacobian matrix of the linearized system of this model has two eigenvalues with the absolute value that is less than 1. That is, they demonstrate that the steady-state point can become a sink, for certain values of ε and η. This result can be summarized as follows.

PROPOSITION 4.1. *The steady state of a neoclassical model characterized by* (4.1) *and* (4.2) *becomes a sink, if the scale of externalities (ε and η) takes certain values.*

If no externalities exist ($\varepsilon = \eta = 0$) in the Benhabib and Farmer(1994) model, it degenerates to the typical neoclassical model. However, the existence of externalities of capital and labor, and endogenous labor supply make the steady state a sink.[5]

Similar problems of stability and indeterminacy can arise in models with an inelastic labor supply.

If the steady state is a sink, candidates for the initial value of $\hat{C}_0$ in (3.4) exist innumerably and cannot be determined uniquely. For this reason, the Neoclassicals bring up the idea of "indeterminacy." The idea of indeterminacy has a significant difficulty from the standpoint of the traditional stability concept, which is used in mathematics. According to the concept, a sink should be regarded as stable, becuase agents can *arbitrarily* choose $\hat{C}_0$ in (3.4).

4.2. Externalities and growth of labor supply. We deal with a continuous-time neoclassical model. The results obtained here are independent of the descriptive method: differential equations or difference equations. We make some notes on Futagami and Mino(1993) at the end of this section.

Households. Let the utility function of households be expressed as

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \tag{4.3}$$

where c denotes per-capita consumption, and σ (> 0) denotes the elasticity of the marginal utility of consumption ($-u'(c)/u''(c)c$).

Let L, w, K and r denote labor forces supplied to firms, the real wage rate, capital stock and the real interest rate, respectively. Then, one has

$$S = wL + rK - C, \tag{4.4}$$

[5] If we assume that the volume of labor supply is constant and only the externalities exist, then, depending on the scale of externalities, the steady state can become a source (Tsuduki, 2010, Ch.2).

where S and C denote savings and consumption, respectively.

For simplicity, we assume no capital depreciation:

$$I = \dot{K}. \tag{4.5}$$

Let n denote the rate of growth of labor force:

$$\dot{L} = nL. \tag{4.6}$$

Goods-market equilibrium is expressed by

$$I = S. \tag{4.7}$$

(4.4)–(4.7) are reduced to the dynamic equation with respect to per-capita capital $k \equiv K/L$:

$$\dot{k} = w + rk - c - nk. \tag{4.8}$$

The utility maximization problem of households becomes:

$$\begin{aligned} &\max_{c} && \int_0^\infty \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \\ &\text{subject to} && \dot{k} = w + rk - c - nk, \end{aligned} \tag{4.9}$$

where ρ (> 0) is the subjective discount rate. Let $\mathcal{H}$ and μ denote the Hamiltonian and the costate variable (see Appendix A), respectively, and the first-order conditions for optimality are given by

$$\frac{\partial \mathcal{H}}{\partial c} = c^{-\sigma} - \mu = 0, \tag{4.10}$$

$$\dot{\mu} = \rho\mu - \frac{\partial \mathcal{H}}{\partial k} = \rho\mu - \mu(r - n). \tag{4.11}$$

The transversality condition is given by

$$\lim_{t \to \infty} (\mu e^{-\rho t} k) = 0. \tag{4.12}$$

Firms. The production function of firms is represented by

$$Y = F(A(\bar{k})K, L), \tag{4.13}$$

where Y, $\bar{k}$ and $A(\bar{k})$ stand for the output, the average economy-wide value of k and the production technology, which is treated as externality by firms, respectively. That technology is a function of capital is based on Arrow's(1962) assumption of "learning by doing."

We assume that a negative externality emerges in the intermediate region of k. The ground of this assumption is that negative externalities, such as congestion and pollution, are rarely observed in the earlier stage of development. However, they become outstanding in the middle stage, and be rarely observed again in the highly developed stage. For a technical reason, we assume

$$\exists k_0 \text{ and } k_1 \text{ such that } \forall k \in (k_0, k_1), \quad A'(k) < 0.$$

Assume that $F(\cdot)$ is of constant returns to scale, and we have:

$$y = f(A(\bar{k})k),$$

where $y \equiv Y/L$ is the per-capita output. $f(\cdot)$ satisfies the following so-called neoclassical features: $\forall x \in \mathbb{R}_+, \quad f'(x) > 0, \ f''(x) < 0, \ \lim_{x\to 0} f'(x) = \infty, \ \lim_{x\to\infty} f'(x) = 0$.

Profit Π is now expressed by

$$\Pi = L[f(A(\bar{k})k) - rk - w]. \tag{4.14}$$

Under perfect competition, the necessary condition for maximizing profit becomes:

$$f'(A(\bar{k})k)A(k) = r. \tag{4.15}$$

The sufficient condition for maximizing profit is $f''(A(\bar{k}))A(\bar{k})^2 < 0$. In equilibrium, $\bar{k} = k$.

In addition, in equilibrium, we obtain the zero-profit condition

$$f(A(k)k) - f'(A(k)k)A(k)k = w. \tag{4.16}$$

Steady state. The system of (4.8), (4.10), (4.11), (4.15) and (4.16) is reduced to the two-dimensional system of differential equations:

$$\dot{c} = \left[f'(A(k)k)A(k) - \rho - n\right]\frac{c}{\sigma}, \tag{4.17a}$$

$$\dot{k} = f(A(k)k) - c - nk. \tag{4.17b}$$

The steady-state (c, k) is the solution of the following system of equations:

$$f'(A(k)k)A(k) = \rho + n, \tag{4.18a}$$

$$c = f(A(k)k) - nk. \tag{4.18b}$$

We obtain

$$\varphi'(k) = f''(A(k)k)\{A'(k)k + A(k)\}A(k) + f'(A(k)k)A'(k), \tag{4.19}$$

where $\varphi(k) \equiv f'(A(k)k)A(k)$.

When $A'(k) > 0$, the sign of the first term of (4.19) is negative and that of the second term is positive. On the other hand, when $A'(k) < 0$, the sign of the first term of (4.19) is positive if $|A'(k)k| > A(k)$, and it is negative if $|A'(k)k| < A(k)$. The sign of the second term is negative.

We assume the following:

ASSUMPTION 1. $\exists k \in (k_0, k_1) \quad \varphi'(k) > 0$.

Under Assumption 1, the function $\varphi(k)$ is drawn as in Fig. 3. (See Appendix A.)

ASSUMPTION 2. $\underline{\rho} < \rho < \overline{\rho}$.

The values of $\underline{\rho}$ and $\overline{\rho}$ are the lower and the upper bound of ρ, respectively (see Fig. 3). This assumption assures the existence of multiple steady states.[6]

Under Assumptions 1 and 2, $\varphi(k)$ and $\rho+n$, which are the both sides of (4.18a), are drawn as shown in Fig. 4.

The steady-states k^*, k^{**} and k^{***}, are determined independent of c. Hence, three vertical lines ($\dot{c} = 0$) are drawn on the k-c plane, and under Assumption 1, (4.18b) ($\dot{k} = 0$) is drawn on the k-c plane, as shown in Fig. 5.

Thus, we obtain the phase diagram of (4.17). Fig. 5 illustrates that (4.17) has three steady-state points: E^*, E^{**}, and E^{***}.

Linearization. The Jacobian matrix of (4.17) is given by

$$J_1 = \begin{bmatrix} 0 & \dfrac{c}{\sigma}\varphi'(k) \\ -1 & \rho + \theta \end{bmatrix},$$

where $\theta \equiv f'(A(k)k)A'(k)k$, all evaluated at the steady state.

At E^* and E^{***}, $A'(k) > 0$ and $\theta > 0$; at E^{**}, $A'(k) < 0$ and $\theta < 0$.

The characteristic equation of matrix J_1 is given as

$$\psi(\lambda) = \lambda^2 - (\text{trace } J_1)\,\lambda + |J_1| = 0, \tag{4.20}$$

where

$$|J_1| = \frac{c}{\sigma}\varphi'(k), \tag{4.21}$$

$$\text{trace } J_1 = \rho + \theta. \tag{4.22}$$

Dynamic property of E^* and E^{***}. At E^* and E^{***}, $\varphi'(k) < 0$. Then, $|J_1| < 0$ from (4.21). Hence, (4.20) has two real roots with opposite signs. Thus, we obtain:

PROPOSITION 4.2. *Under Assumptions* 1 *and* 2*, E^* and E^{***} are saddle points.*

Dynamic property of E^{**}. At E^{**}, $\varphi'(k) > 0$. Then, $|J_1| > 0$ from (4.21). Further, from (4.22), we obtain

$$\text{trace } J_1 > 0, \ \text{ if } \rho > -\theta\,; \quad \text{trace } J_1 < 0, \ \text{ if } \rho < -\theta. \tag{4.23}$$

[6] Tsuduki(2010) provides an analysis of cases where ρ does not lie in the range $\underline{\rho} < \rho < \overline{\rho}$.

Hence, if $\rho > -\theta$, (4.20) has two roots with a positive real part and the steady state is a source; if $\rho < -\theta$, (4.20) has two roots with negative real parts and the steady state is a sink.

Moreover, we can examine whether roots are real or complex by checking the sign of the discriminant D of (4.20);

$$D = (\text{trace } J_1)^2 - 4|J_1|.$$

If $D \geq 0$, two roots are real numbers and the steady state is a node; if $D < 0$, two roots are conjugated complex numbers and the steady state is a spiral.

We set the following assumption.

ASSUMPTION 3. *There exists a constant* $\Omega > 0$ *such that*

$$D \geq 0 \text{ if } |\rho + \theta| \geq \Omega, \text{ and } D < 0 \text{ if } |\rho + \theta| < \Omega.$$

From this assumption and (4.23), we have:

$$\begin{cases} \text{Stable node} & \text{if} \quad \rho \leq -\theta - \Omega, \\ \text{Stable spiral} & \text{if} \quad -\theta - \Omega < \rho < -\theta, \\ \text{Unstable spiral} & \text{if} \quad -\theta + \Omega > \rho > -\theta, \\ \text{Unstable node} & \text{if} \quad \rho \geq -\theta + \Omega. \end{cases} \tag{4.24}$$

Fig. 2 is a diagram for (4.24). It clearly shows that the larger the scale of negative externality of capital (represented by $|\theta|$) and the smaller the household's discount rate (represented by ρ) are, the more stable the steady state will be.

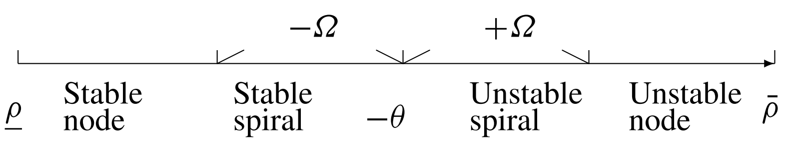

FIGURE 2. θ, ρ, and dynamic property of E^{**}

Further, applying the Hopf bifurcation theorem by Asada(1995), if $\rho = \hat{\rho} \equiv -\theta$, the following conditions hold:

(1) $-\text{trace } J_1(\hat{\rho}) = -(\hat{\rho} + \theta) = 0$, $|J_1|(\hat{\rho}) = c\phi'(k)/\sigma > 0$,

(2) $d(-\text{trace } J_1(\hat{\rho}))/d\rho = 1 \neq 0$.

Hence, the conditions for a Hopf bifurcation are satisfied when $\rho = \hat{\rho}$ and cycles exist in the neighborhood of $\hat{\rho}$.

If the bifurcation is supercritical, a stable cycle exists in a range of $-\theta + \Omega > \rho > -\theta$; that is, a solution starting from an arbitrary point, except for the one from the steady state, will converge to cycle. On the other hand, if the bifurcation is subcritical, an unstable cycle exists in the range of $-\theta - \Omega < \rho < -\theta$; that is, a solution starting from arbitrary point inside the cycle will converge to it, whereas a solution starting from arbitrary point outside the cycle will diverge from the steady state. These results can be summarized as follows:

PROPOSITION 4.3. *Under Assumptions 1–3, the steady state E^{**} is a stable node for $\rho \leq -\theta - \Omega$, and an unstable node for $\rho \geq -\theta + \Omega$. In addition, E^{**} is a stable spiral for $-\theta - \Omega < \rho < -\theta$, and an unstable spiral for $-\theta + \Omega > \rho > -\theta$. A Hopf bifurcation occurs at $\rho = -\theta$.*

Propositions 4.2 and 4.3 suggest the following. In economies located near the lower state (E^*) and the higher state (E^{***}), externalities do not affect the stability. However, in economies located near the middle state (E^{**}), negative externality of capital contributes to stabilize an economy: the larger the scale of externalities, more stable an economy. (This is valid irrespective of the type of a Hopf bifurcation.)

Notes on Futagami-Mino(1993). Futagami and Mino(1993), (in what follows, we refer FM) proposed a very similar model with ours. The main differences between these models are summarized as follows:

(a) FM assume the production function of the form $y = A(k)f(k)$; we assume the form $y = f(A(k)k)$.
(b) FM only consider a positive externality (that is, $A'(k) > 0$); we allow an externality to become negative (that is, $A'(k) < 0$).
(c) To produce an S-shape of $\varphi(k)$, FM assume the existence of a threshold of externalities, whereas this study demonstrated that a negative externality of capital can also generate an S-shape of $\varphi(k)$.
(d) FM's analysis is made mainly by utilizing figures, but our analysis is more analytical. For example, FM show the existence of a limit cycle by applying the Poincare-Bendixson theorem. However, we used the Hopf bifurcation theorem to demonstrate that. Moreover, FM discuss only the case in which the middle-state equilibrium is unstable; however, we also consider cases in which it is stable.

5. Concluding Remarks

So far, we took up the Kaldor model, the simplest neoclassical macrodynamic model and two Ramsey-type models with externalities. We sum up our discussion here.

In the first half of this article, we discussed the typical Post-Keynesian model and neoclassical macrodynamic model.

First, we can confirm that the concept of stability of equilibrium employed by Post-Keynesians is mathematically natural one.

Second, the point that we make in this study is the feasibility of equilibrium; whether equilibrium is reachable. The neoclassical claim of stability of the saddle point equilibrium is found unsupportable, because the choice of initial points is not adequate. The growth path appears to be oriented to a steady state at the onset, however, it proceeds to a doom at the end. These models ironically explain a single-shot crash, despite the Neoclassicals intentions. In addition, the stability concept of indeterminacy is also inadequate because that situation should be regarded as stable, as is the way with mathematics.

In the latter half of this article, we dealt with two examples with externalities, which are not accompanied by saddle points. Benhabib-Farmer(1994) first built a Ramsey-type model with externalities without the saddle point, but they interpret their result negatively: they thought externalities have the potential to destabilize an economy because they cause indeterminacy. They got blind because of their strong neoclassical prejudice.

We argued that the following assumptions adopted by Neoclassicals are unreasonable: (i) households can know the location of the saddle path and exactly jump to that; (ii) stable solutions imply indeterminacy of equilibrium. We also argued that Keynesian approaches can avoid these unnatural assuptions and generate a range of dynamic outcomes.

We pointed out that introduction of ideas that came up with market failures seems to expel saddle points from the system.

Hence, in the *third* place, we can state the following: noting that the saddle point is equipped with the structural stability (Lorenz, 1993, Ch.3), which implies that chages in the values of parameters do not affect the dynamic property of the steady state. However, that the introduction of such factors as externalities alters the nature of equilibrium is a remarkable fact; that is, defining the perfect market mechanism as referring to the feature of an economy with no perturbing factors, the perfect market mechanism seems to be accompanied by saddle points.

Appendix A. Externalities and Growth of Labor Supply

Hamiltonian. The Hamiltonian function of the problem is defined as

$$\mathcal{H} = \frac{c^{1-\sigma} - 1}{1-\sigma} + \mu[w + rk - c - nk],$$

where μ is the costate variable of the state variable k.

Figures.

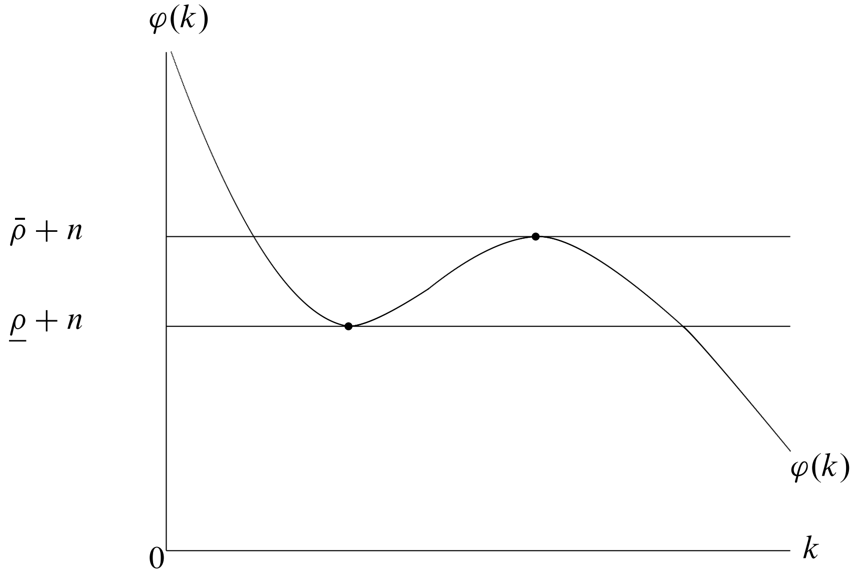

FIGURE 3. Upper and lower bounds of ρ

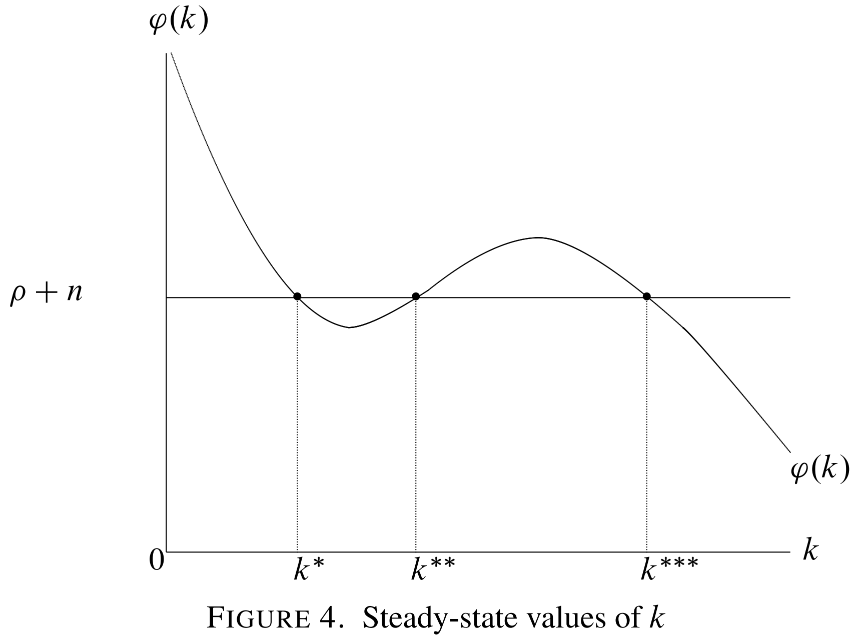

FIGURE 4. Steady-state values of k

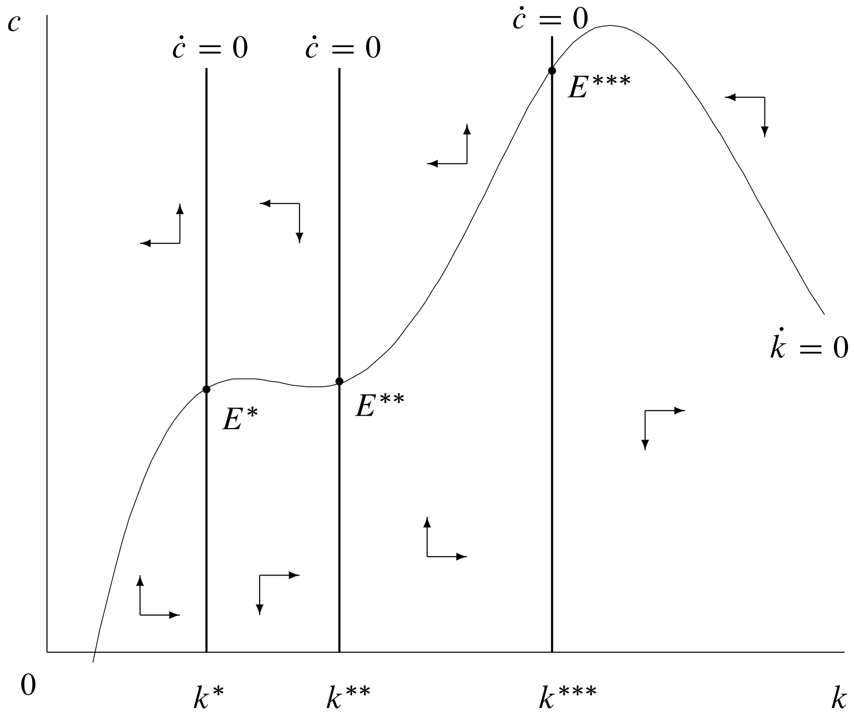

FIGURE 5. Phase diagram

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