Post Keynesian Review
Vol. 2, No. 2, Pages 19–30 (the 30th October, 2014)
ISSN 2187-4875

Did Sraffa Succeed in Proving the Perron-Frobenius Theorem?

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ABSTRACT. In attempting to verify the existence of Sraffa's Standard commodity and Standard ratio, anyone accustomed to an applied linear algebra resorts to the Perron-Frobenius theorem. However, when Sraffa outlined his theory, he did not know the theorem, indicating that he presented an original proof for the Perron-Frobenius theorem to validate his theory. The following question arises: Is Sraffa's proof correct or false? This study attempts to address this problem and finds that Sraffa's proof is partly correct: His proof is applicable only to Perron's theorem but not to Frobenius' theorem. It therefore requires modification: this study aims at achieving this modification, thus extending its applicability to both theorems.

***Keywords*:** Sraffa; Standard commodity; Standard ratio; Perron-Frobenius theorem.

1. Introduction

On the existence of the Standard commodity, Piero Sraffa remarked as follows:

> **36** In the following five sections it is sought to prove that there always is a way, and never more than one way, of transforming a given economic system into a Standard system: in other words, that there always is one, and only one, set of multipliers which, if applied to the several equations or industries composing the system, will have the effect of rearranging them in such proportions that the commodity-composition of the aggregate means of production and that of the aggregate product are identical. (Sraffa, 1960, p.26.)

Prior to the argument of the Standard system, Sraffa supposed a new system of equations named as the q-system:

$$(1.1) \qquad (1+R)\sum_{j=1}^{n} A_{ij}q_j = x_i q_i, \quad i = 1, 2, \ldots, n,$$

where $A_{ij} \geq 0$ represents the quantity of commodity i required to produce the commodity j, x_i represents the produced amount of commodity i, R represents

First draft received: the 7th October, 2014. The final accepted: the 30th, October, 2014.

the Standard ratio and q_i represents the multipliers, Then, Sraffa maintained that the problem of constructing the Standard system amounted to finding positive q_is and positive R in the q-system. This implies that Sraffa confronted a mathematical problem: any matrix $A = (A_{ij}) \geq O$ has a positive eigenvalue and a positive eigenvalue associated with it. This problem was, however, already solved by two German mathematicians, Oscar Perron and Ferdinand G. Frobenius. Perron(1907) dealt with positive matrices, and Frobenius(1912) extended Perron's contribution to the non-negative matrices. Today, their results are called the Perron-Frobenius theorem. Approximately 50 years later, Sraffa, who did not know of the preceding studies, worked independently on this problem under the necessity of proving the existence of the Standard commodity and the Standard ratio.[1] Therefore, an interesting question arises: Did Sraffa succeeded in proving the Perron-Frobenius theorem in Section 37 of his book? Recently, two economists investigated this problem. Lippi(2008,p.247) concluded that "Sraffa does not provide a rigorous proof." Subsequent to Lippi, Salvadori(2008, p.253) analyzed the issue more generally and arrived at a similar conclusion: "[T]he algorithm in Section 37 of Sraffa's book is not precisely stated and that it does not need to converge to the desired eigenvalue and eigenvector."[2]

Although the author largely agree with the conclusions of these two economists, one aspect of their research is questionable. Lippi and Salvadori applied Section 37 only to Frobenius's theorem but not to Perron's theorem: thus, the problem remains as to whether Sraffa succeeded in proving Perron's theorem. In this study we examine Section 37 and apply Sraffa's proof to Perron's theorem, which treats positive matrices, and then to Frobenius' theorem, which deals with nonnegative matrices.

This paper is organized as follows. In Section 2, we present Section 37 of Sraffa's book in its original form to provide a background. In Section 3, we translate Sraffa's non-mathematical explanations into mathematical terms and investigate whether Sraffa's proof is applicable to the Perron theorem. In Section 4, we address Frobenius' theorem and verify it by adding a few propositions to Perron's theorem. The last section presents the conclusions.

[1] According to the extrapolation the latest research on Sraffa's unpublished papers and correspondence, the Majorca draft was written in 1955, and the argument for the existence of the Standard commodity did not exist in this draft. Therefore, the draft of section 36 was presumably written after 1955. For more details, see Matsumoto(2010) and Fujii(2012).

[2] See also Salvadori(2011) and Kurz and Salvadori(2007).

2. Section 37

Section 37 of Sraffa(1960, pp.26-7) runs as follows:

> **37** That any actual economic system of the type we have been considering can always be transformed into a Standard system may be shown by an imaginary experiment.
>
> (The experiment involves two types of alternating steps. One type consists in changing the proportions of the industries; the other in reducing in the same ratio the quantities produced by all industries, while leaving unchanged the quantities produced by all industries, while leaving unchanged the quantities used as means of production.)
>
> We start by adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement.
>
> Let us next imagine gradually to reduce by means of successive small proportionate cuts the product of all the industries, without interfering with the quantities of labour and means of production that they employ.
>
> As soon as the cuts reduce the production of any one commodity to the minimum level required for replacement, we readjust the proportions of the industries so that there should again be a surplus of each product (while keeping constant the quantity of labour employed in the aggregate). This is always feasible so long as there is a surplus of some commodities and a deficit of none.
>
> We continue with such an alternation of proportionate cuts with the re-establishment of a surplus for each product until we reach the point where the products have been reduced to such an extent that all-round replacement is just possible without leaving anything as surplus product.
>
> Since to reach this position the products of all the industries have been cut in the same proportion we are now able to restore the original conditions of production by increasing the quantity produced in each industry by a uniform rate; we do not, on the other hand, disturb the proportions to which the industries have been brought. The uniform rate which restores the original conditions of production is R and the proportions attained by the industries are the proportions of the Standard system.

Sraffa gives his idea in everyday language. However, as the aim of our study is to examine whether or not it succeeds in proving the Perron-Frobenius theorem, we need to translate it into mathematical terms and ascertain its plausibility from a mathematical point of view. The next section examines this.

3. Perron's Theorem

We examine the applicability of Sraffa's proof to Perron's theorem (Gantmacher, 1959, p.53).

THEOREM 3.1 (Perron). *Let A be a positive matrix of degree n. Then there exists a pair of $x > 0$ and $\alpha > 0$ such that $Ax = \alpha x$.*

We divide the Section 37 into four parts and examine these parts from a mathematical point of view.[3]

3.1. Step 1.

3.1.1. *Starting point.*

> We start by adjusting the proportions of the industries of the system in such a way that of each basic commodity a larger quantity is produced than is strictly necessary for replacement. (Sraffa, 1960, p.26.)

3.1.2. *Mathematical translation.* Let A be a positive matrix of degree n and x be an n-dimensional vector. When we define x^0 as

$$x^0 = \frac{x}{\|x\|_\infty},$$

where $\|x\|_\infty$ denotes the ∞-norm of x, there exists a pair of $a > 0$ and $b > 0$ such that

$$ax^0 < Ax^0 < bx^0. \tag{3.1}$$

3.1.3. *Remark.*

(1) A is an input matrix, and x^0 is a production vector and normalized as $\mathbf{0} < x^0 \leq \mathbf{1}$, where all entries of $\mathbf{1}$ are unity.
(2) $A > O$, and the system represented by A consists of only basic commodities.

[3] Our main source on the Perron-Frobenius theorem are Gantmacher(1959), and Nikaido(1968). See also Bellman(1970) and Meyer(2000).

(3) Sraffa's statement “the proportions of the industries of the system in such a way that ... a larger quantity is produced than is strictly necessary for replacement" is mathematically translated as follows: We multiply x^0 by $b\left(> \max_{1\le i\le n} \frac{\sum_{j=1}^n a_{ij}x_j^0}{x_i^0}\right)$ and obtain a strict inequality $Ax^0 < bx^0$. The set bx^0 is the upper bounds of Ax^0.

(4) Although Sraffa does not consider the lower bounds, we define them here for convenience; if we multiply x^0 by $a\left(< \min_{1\le i\le n} \frac{\sum_{j=1}^n a_{ij}x_j^0}{x_i^0}\right)$, we get a strict inequality $ax^0 < Ax^0$. The set ax^0 gives the lower bounds of Ax^0.

3.2. Step 2.

3.2.1. ***Proportionate Cuts.***

> Let us next imagine gradually to reduce by means of successive small proportionate cuts the product of all the industries, without interfering with the quantities of labour and means of production that they employ.
> . . . [T]he cuts reduce the production of any one commodity to the minimum level required for replacement[.] (Sraffa, 1960, p.26.)

3.2.2. *Mathematical translation.* We define α_1 and β_1 as

$$\alpha_1 = \min_{1\le i\le n} \frac{(Ax^0)_i}{x_i^0}, \qquad \beta_1 = \max_{1\le i\le n} \frac{(Ax^0)_i}{x_i^0},$$

where $(Ax^0)_i = \sum_{j=1}^n a_{ij}x_j^0, i = 1, 2, \ldots n$. We obtain

$$\alpha_1 x^0 \le Ax^0 \le \beta_1 x^0. \tag{3.2}$$

3.2.3. ***Remark.***

(1) Sraffa wrote, “without interfering with the quantities of labour." However, the quantities of labour are not related to the proof; therefore, we do not consider this paragraph.

(2) This paragraph states that quantities of products are reduced without changing the labour quantities and the means of production. An individual familiar with ordinary production theory would conceive of this as impossible. However, as Sraffa previously remarked, we examine “an imaginary experiments," thus allowing for the possibility to be considered.

(3) Let $\mathrm{I} \equiv \{1, 2, \ldots, n\}$ denote the index of industries. We divide this set into two subsets, M and N, such that

$$\mathrm{M} = \Big\{i \in \mathrm{I} \,\Big|\, \sum_{j=1}^{n} a_{ij} x_j^0 = \beta_1 x_i^0\Big\}, \quad \mathrm{N} = \Big\{i \in \mathrm{I} \,\Big|\, \sum_{j=1}^{n} a_{ij} x_j^0 < \beta_1 x_i^0\Big\}.$$

The system of inequalities (3.2) implies that M is not empty. Then, the postulation "[T]he cuts reduce the production of any one commodity to the minimum level required for replacement" is represented.

3.3. Step3.

3.3.1. *Readjustment of surplus.*

> As soon as the cuts reduce the production of any one commodity to the minimum level required for replacement, we readjust the proportions of the industries so that there should again be a surplus of each product (while keeping constant the quantity of labour employed in the aggregate). This is always feasible so long as there is a surplus of some commodities and a deficit of none. (Sraffa, 1960, p.26).

3.3.2. *Mathematical translation.* Let $C = \dfrac{A}{\|Ax^0\|_\infty}$. We premultiply the inequality (3.2) by C. As C is a positive matrix, we get:[4]

$$\alpha_1 C x^0 < CAx^0 < \beta_1 C x^0. \tag{3.3}$$

Now, let $x^1 = Cx^0$, and this is rewritten as follows;

$$\alpha_1 x^1 < Ax^1 < \beta_1 x^1. \tag{3.4}$$

3.3.3. *Remark.*

(1) We multiply the inequality (3.2) by a positive matrix C and "readjust the proportions of the industries so that there should again be a surplus of each product."

(2) We transformed the weak inequality $Ax^0 \leq \beta_1 x^0$ into the strict inequality $Ax^1 < \beta_1 x^1$ by multiplying it by a positive matrix C. This procedure reflects Sraffa's statement: "[t]his is always feasible so long as there is a surplus of some commodities and a deficit of none."

(3) In this way we returned to the situation of (3.1), in which the vector x^1 is normalized by a ∞-norm to satisfy the condition $x^1 \leq \mathbf{1}$.

3.4. Step 4.

[4] Let P be a positive matrix and y be a nonnegative vector, and Py is positive.

3.4.1. ***Convergence.***

> We continue with such an alternation of proportionate cuts with the re-establishment of a surplus for each product until we reach the point where the products have been reduced to such an extent that all-round replacement is just possible without leaving anything as surplus product. (Sraffa, 1960, p.27).

3.4.2. ***Mathematical translation.*** In the same manner as that of Step 2, we define α_2 and β_2 as

$$\alpha_2 = \min_{1\le i\le n} \frac{(Ax^1)_i}{x_i^1}, \quad \beta_2 = \max_{1\le i\le n} \frac{(Ax^1)_i}{x_i^1}.$$

Then, $\alpha_2 x^1 \le Ax^1 \le \beta_2 x^1$ is shown. From the definition of α_2 and β_2, $\beta_2 < \beta_1$ and $\alpha_1 < \alpha_2$ are derived.

By repeating Step 2 and Step 3, we obtain the sequence

$$0 < \alpha_1 < \alpha_2 < \alpha_3 \ldots\ldots < \beta_3 < \beta_2 < \beta_1.$$

$\{\alpha_n\}_{n=1,2,\ldots}$ is strictly monotonically increasing and is bounded above. $\{\beta_n\}_{n=1,2,\ldots}$ is strictly monotonically decreasing and bounded below. Accordingly, there exist limit points;

$$\lim_{n\to\infty} \alpha_n = \alpha, \qquad \lim_{n\to\infty} \beta_n = \beta.$$

3.4.3. ***Remark.*** Sraffa asserted that if "[w]e continue with an alternation of proportionate cuts with the re-establishment of a surplus for each product," the sequence converges on the limit point. Therefore:

$$\lim_{n\to\infty} \|Ax^n - \alpha_n x^n\|_\infty = \|Ax - \alpha x\|_\infty = 0.$$

However, he did not furnish its proof. We attempt to close this gap.

3.4.4. ***Supplement.*** Suppose $\varepsilon = \min_{i,j} a_{ij}$ and $z > \mathbf{0}$. Then, there exists a system of inequalities:

$$\sum_{j=1}^{n} a_{ij} z_j \ge \sum_{j=1}^{n} \varepsilon z_j \ge \varepsilon \|z\|_\infty.$$

If we apply these inequalities to the positive vector $Ax^n - \alpha_n x^n$, we have

$$\sum_{j=1}^{n} a_{ij}(Ax^n - \alpha_n x^n)_j \ge \varepsilon \|Ax^n - \alpha_n x^n\|_\infty \quad (i = 1, \ldots, n), \tag{3.5}$$

where $(Ax^n - \alpha_n x^n)_j$ is the j-th component of the vector $Ax^n - \alpha_n x^n$. If we premultiply both sides of (3.5) by $\dfrac{1}{\|Ax^n\|_\infty}$, we see that, for $i = 1, \ldots, n$,

$$\frac{1}{\|Ax^n\|_\infty} \sum_{j=1}^{n} a_{ij}(Ax^n - \alpha_n x^n)_j \geq \frac{1}{\|Ax^n\|_\infty} \varepsilon \|Ax^n - \alpha_n x^n\|_\infty. \tag{3.6}$$

In view of definition $x^{n+1} = Cx^n$, the left-hand side of (3.6) is transformed as follows: for $i = 1, \ldots, n$,

$$\begin{aligned}\frac{1}{\|Ax^n\|_\infty} \sum_{j=1}^{n} a_{ij}(Ax^n - \alpha_n x^n)_j &= \sum_{j=1}^{n} a_{ij}(Cx^n)_j - \frac{\alpha_n}{\|Ax^n\|_\infty} \sum_{j=1}^{n} a_{ij}\,(x^n)_j \\ &= \sum_{j=1}^{n} a_{ij}\left(x^{n+1}\right)_j - \alpha_n \left(x^{n+1}\right)_i .\end{aligned}$$

Define $\alpha_{n+1} = \max\limits_{1 \leq i \leq n} \dfrac{(Ax^{n+1})_i}{x_i^{n+1}}$, and let k stand for the index of components of x^{n+1} that satisfies the definition of α_{n+1}. Then, we transform the left-hand side of (3.6) with respect to its k-th component as follows:

$$\begin{aligned}\sum_{j=1}^{n} a_{kj}\left(x^{n+1}\right)_j - \alpha_n \left(x^{n+1}\right)_k &= \alpha_{n+1}\left(x^{n+1}\right)_k - \alpha_n \left(x^{n+1}\right)_k \\ &= (\alpha_{n+1} - \alpha_n)\left(x^{n+1}\right)_k .\end{aligned}$$

Concerning the k-th component of the inequality (3.6), we obtain

$$\begin{aligned}(\alpha_{n+1} - \alpha_n)\left(x^{n+1}\right)_k &= \frac{1}{\|Ax^n\|_\infty} \sum_{j=1}^{n} a_{kj}(Ax^n - \alpha_n x^n)_j \\ &\geq \frac{\varepsilon}{\|Ax^n\|_\infty} \|Ax^n - \alpha_n x^n\|_\infty. \end{aligned} \tag{3.7}$$

Since $\|x^{n+1}\|_\infty \leq 1$, this implies

$$\frac{\|Ax^n\|_\infty}{\varepsilon} (\alpha_{n+1} - \alpha_n) \geq \|Ax^n - \alpha_n x^n\|_\infty. \tag{3.8}$$

Besides, $\lim\limits_{n\to\infty} \alpha_n = \alpha$ implies $\lim\limits_{n\to\infty} (\alpha_{n+1} - \alpha_n) = 0$, so that this inequality entails

$$\lim_{n\to\infty} \|Ax^n - \alpha_n x^n\|_\infty = 0. \tag{3.9}$$

Let $Q = \{x \in R^n \,|\, \|x\|_\infty \leq 1\}$. Since $\|x^n\|_\infty \leq 1$, it follows that $\{x^n\} \subset Q$. Q is a compact set, so that there exists a converging subsequence of $\{x^n\}$. Suppose it is $\{x^{n_m}\}_{m=1,2,\ldots}$, and we obtain $\lim\limits_{m\to\infty} x^{n_m} = x \in Q$. Therefore,

$$\lim_{m\to\infty} \|Ax^{n_m} - \alpha_{n_m} x^{n_m}\|_\infty = \|Ax - \alpha x\|_\infty = 0.$$

By the definition of ∞-norm, if $\|z\|_\infty = 0$ then $z = \mathbf{0}$. We obtain

$$Ax = \alpha x. \tag{3.10}$$

$x > \mathbf{0}$ and $\alpha > 0$ are obvious.

3.5. Result. Although Section 37 is not a mathematically rigorous proof for Perron's theorem but just a rough sketch, its algorithm indicates the correct way to achieve the proof. Thus, inferring that Sraffa succeeded in proving Perron's theorem seems correct.

4. Frobenius' Theorem

Let us turn to the next problem, namely, whether or not Section 37 applies to Frobenius' theorem.

THEOREM 4.1 (Frobenius). *Let A be an indecomposable non-negative matrix of degree n. Then there exists a pair of $x > \mathbf{0}$ and λ such that $Ax = \lambda x$.*

4.1. Sraffa's proof. Since C is a normalized matrix obtained from A, $A \geq O$ implies $C \geq O$. Therefore, we cannot apply Sraffa's approach to Frobenius' theorem. For example, let

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ and } \delta = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix},$$

and, subsequently, we have $\gamma \leq \delta$. Premultiplying both sides by $C \geq O$, we obtain $C\gamma \leq C\delta$. In this example we cannot derive $C\gamma < C\delta$; hence, inequality (3.4) cannot be derived, either; this in turn, means that we cannot return to Step 1. Sraffa's algorithm cannot be extended further and the proof fails.

4.2. Proof. The above counterexample shows that Sraffa's explanation cannot be applied to the Frobenius theorem as it is; therefore, we attempt to find another way to prove the theorem on the basis of the obtained result. This task will be made in Appendix.

5. Concluding Remarks

Sraffa succeeded in proving Perron's theorem but not the Frobenius' theorem. Let us examine why this result occurred.

Whichever theorem is examined, the core of the proof is to establish a sequence: $0 < \alpha_1 < \alpha_2 < \alpha_3 \ldots\ldots < \beta_3 < \beta_2 < \beta_1$. Sraffa derives this sequence from his original algorithm, to use our terms, Step 2 (Proportionate cut) and Step 3 (Readjustment of surplus). However, his algorithm presents a problem: it works well if a matrix is positive, whilst it works poorly if a matrix is non-negative.

Lippi(2008) and Salvadori(2008, 2011) identified a defect in the algorithm, and attempted to correct it. Salvadori(2008, 2011) showed a non-converging example in the case of a non-negative matrix. They showed that if the algorithm is revised correctly, it converges to the Frobenius root and the Frobenius vector. In contrast to their work, we showed that Sraffa's algorithm is workable—not on Frobenius' theorem but on Perron's theorem. Moreover, if the latter is premised, the former is proven by adding a few propositions to the latter.

In summary, *our major findings* are as follows: 1) *Sraffa succeeded in proving Perron's theorem.* 2) Section 37 *cannot be applied to Frobenius' theorem.* 3) *If Perron's theorem is assumed, then Frobenius' theorem is proven by adding new conditions to the former*: *but, as Sraffa did not mention them, we conclude that Sraffa fails to prove Frobenius' theorem.*

Appendix A. Proof of Frobenius' Theorem

We start from the following lemma (Nikaido, 1968, p.103).

LEMMA A.1. *Let A and B be positive matrices, and λ^A and λ^B the Perron root of A and B respectively. If $A \geq B$, then $\lambda^A \geq \lambda^B$.*

For any non-negative matrix, the following theorem is established.[5]

THEOREM A.2. *Let A be a non-negative matrix of degree n. There exists a pair of $x \geq \mathbf{0}$ and $\alpha \geqq 0$ such that $Ax = \alpha x$.*

Proof. Replace the zero components of non-negative matrix A with $1/t$ (where t is a positive integer), and let the replaced matrix be $A^{(t)}$. As $A^{(t)} > O$, there exists a pair of the Perron root $\lambda^{(t)} > 0$ and the Perron vector $x^{(t)} > \mathbf{0}$ such that

$$A^{(t)}x^{(t)} = \lambda^{(t)}x^{(t)}, \qquad t = 1, 2, \ldots \tag{A.1}$$

from Perron's theorem. Let the sum of the components of $x^{(t)}$ be unity; namely,

$$S_n = \left\{x \mid \sum_{j=1}^{n} x_i = 1, x \geq \mathbf{0}\right\},$$

and $x^{(t)} \in S_n$ (t=1,2,...) follows. Since $A^{(t)}$ is a continuous function of t, $\lim_{t\to\infty} A^{(t)} = A$.

[5]The proof is based on literature listed in Gantmacher(1959) and Nikaido(1968).

Next, if we show that $\lambda^{(t)}$ and $x^{(t)}$ converge to a non-negative eigenvalue and a non-negative eigenvector respectively, the proof will be completed.

By the definition of $A^{(t)}$, it follows that $A^{(t)} > A^{(t+1)} > 0$ ($t = 1, 2, \ldots$). Then $\lambda^{(t)} > \lambda^{(t+1)} > 0$ ($t = 1, 2, \ldots$) by Lemma A.1. Since $\{\lambda^{(t)}\}_{t=1,2,\ldots}$ are monotone decreasing sequences, they are bounded below and have the limit point: $\lim_{t\to\infty} \lambda^{(t)} = \lambda^0$.

Since S_n is compact, a converging subsequence exists. We pick up a strictly monotone increasing sequence, that is, $x^{(\upsilon_1)}, x^{(\upsilon_2)}, x^{(\upsilon_3)} \ldots$ ($\upsilon_1 < \upsilon_2 < \upsilon_3 \ldots$) and let $\lim_{k\to\infty} x^{(\upsilon_k)} = x^0$.

From (A.1), it follows that

$$A^{(\upsilon_k)}x^{(\upsilon_k)} = \lambda^{(\upsilon_k)}x^{(\upsilon_k)}, \qquad k = 1, 2, \ldots \tag{A.2}$$

Suppose $k \to \infty$, and we obtain:

$$A^{(\upsilon_k)} \to A, \quad \lambda^{(\upsilon_k)} \to \lambda^0, \quad x^{(\upsilon_k)} \to x^0.$$

Namely, we have

$$Ax^0 = \lambda^0 x^0, \tag{A.3}$$

where since $x^{(t)} \in S_n$, $x^0 \geq 0$. $\lambda^0 \geqq 0$ is obvious. □

We proved the non-negativity. Next, if the indecomposability of the non-negative matrix is assumed, the existence of a positive eigenvalue and a positive eigenvector is proven with it. The next lemma will be applied. (Gantmacher, 1959, p.51.)

LEMMA A.3. *Let A be a non-negative indecomposable matrix of degree n. For any non-negative vector $x \geq \mathbf{0}$, we have $(I + A)^{n-1}x > \mathbf{0}$.*

Now, we prove Theorem 4.1 (Frobenius).

Proof. From Theorem 4.3 non-negativity is established, but proof of positivity remains. If we add x^0 to both sides of (A.3), we obtain

$$(I + A)x^0 = (1 + \lambda^0)x^0. \tag{A.4}$$

Premultiply both sides of (A.4) by $I + A$ and iterate this procedure. We obtain

$$(I + A)^{n-1}x^0 = (1 + \lambda^0)^{n-1}x^0 \tag{A.5}$$

from (A.4). The left-hand side of (A.5) is positive in view of Lemma A.3, and $(1 + \lambda^0)^{n-1} > 0$; hence, we have $x^0 > \mathbf{0}$. If $x^0 > \mathbf{0}$ and $\lambda^0 \neq 0$, then $\lambda^0 > 0$. □

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