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An Analytical Critique of 'New Keynesian' Dynamic Model

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ABSTRACT. In this paper, we present an analytical critique of New Keynesian dynamic model. It is shown analytically that the prototype New Keynesian dynamic model produces the paradoxical behaviors that are inconsistent with the empirical facts. We also present more traditional alternative approach that is consistent with the empirical facts, which is called the Old Keynesian dynamic model.

***Keywords*:** New Keynesian dynamic model, (in)determinacy, exogenous disturbance, Old Keynesian dynamic model, (in)stability, Hopf bifurcation, endogenous fluctuation.

***JEL Classification*:** E12, E31, E32, E52

1. Introduction

Recently, the so-called New Keynesian(NK) dynamic model, which is represented by Woodford(2003), Bénassy(2007), Galí(2008) and others, is becoming more and more influential in the mainstream macroeconomic analysis. The method of modeling that is adopted in this approach is based on the dynamic optimization of the *representative agents* with perfect foresight or rational expectations, but it contradicts Keynes'(1936) own vision on the working of the market economy that is based on the *bounded rationality* due to the inherent *fundamental uncertainty*.

In this paper, we present an internal and analytical critique of the NK dynamic model. The adjectives *internal* and *analytical* mean that our critique is based on the detailed mathematical study of the prototype NK dynamic model. It is shown that this model produces the paradoxical behaviors of the main variables which contradict the empirical facts. We also propose an alternative approach that is called the *Old Keynesian* dynamic model, which is immune from such kind of anomaly.

In Section 2, we formulate the simplest version of the prototype NK dynamic model in terms of a linear system of difference equations. In Section 3, we present a detailed mathematical analysis of such a model without exogenous disturbances. Section 4 is devoted to critical considerations of the prototype NK dynamic model,

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which is largely based on the analysis in Section 3. Section 5 provides a detailed analysis of the effects of the exogenous but non-stochastic disturbances in the NK setting.

The considerations up to Section 5 have rather critical and negative implications, although they are largely based on the detailed mathematical analyses. In Section 6, however, we propose an alternative positive and constructive approach, which is called the Old Keynesian dynamic model in contrast to the NK dynamic model. Section 7 is devoted to concluding remarks. Mathematical appendices supplement the descriptions in the text.

2. Formulation of the Prototype NK Dynamic Model

The simplest version of the prototype NK dynamic model may be formulated as follows.[1]

$$\pi_t = \pi^e_{t+1} + \alpha(y_t - \bar{y}) + \varepsilon_t, \tag{2.1}$$

$$y_t = y^e_{t+1} - \beta(r_t - \pi^e_{t+1} - \rho_o) + \xi_t, \tag{2.2}$$

$$r_t = \rho_o + \bar{\pi} + \gamma_1(\pi_t - \bar{\pi}) + \gamma_2(y_t - \bar{y}), \tag{2.3}$$

$$\pi^e_{t+1} = E_t\pi_{t+1}, \quad y^e_{t+1} = E_t y_{t+1}, \tag{2.4}$$

where π_t the rate of price inflation at period t, π^e_{t+1} expected rate of price inflation at period $t+1$, *i.e.* rate of price inflation at period $t+1$ that is expected at period t, Y_t real national income at period t, $y_t = \log Y_t$, Y^e_{t+1} expected real national income at period $t+1$ (real national income at period $t+1$ that is expected at period t), $y^e_{t+1} = \log Y^e_{t+1}$, r_t nominal rate of interest at period t, ρ_o equilibrium value of the real rate of interest (fixed), $\bar{Y}$ equilibrium value of the real national income (fixed), $\bar{y} = \log \bar{Y}$, and $\bar{\pi}$ target rate of inflation that is set by the central bank (fixed). All of the parameter values α, β, γ_1 and γ_2 are positive. The terms ε_t and ξ_t represent the (stochastic or non-stochastic) exogenous disturbances. Note that $r_t - \pi^e_{t+1}$ expresses the expected real rate of interest at period t.

Equations (2.1) and (2.2) represent *NK Phillips curve* and *NK IS curve* respectively. According to the NK convention that sticks to the so-called *microeconomic foundation*, (2.1) is derived from the optimizing behavior of the imperfectly competitive firms with costly price changes, and (2.2) is derived from the first order

[1] As for the textbooks of the New Keynesian dynamic model by the original authors, see Woodford(2003), Bénassy(2007) and Galí(2008). Romer(2006) contains some useful expositions. The formulation in this section is the simplified version of the formulation by Asada, Chiarella, Flaschel and Proaño(2007). See also Flaschel, Franke and Proaño(2008) and Asada, Chiarella, Flaschel and Franke(2010) as for similar formulations.

condition of the inter-temporal optimization for the representative consumers, that is called the *Euler equation* of consumption.[2] This means that the real aggregate demand becomes a decreasing function of the real rate of interest even if the firms' investment expenditure is neglected in this model. (2.3) represents the interest rate monetary policy rule of the central bank by means of the *Taylor rule*, which is also called the *flexible inflation targeting* that considers both of the rate of price inflation and the employment(real output).[3] (2.4) is the formalization of the *rational expectation* hypothesis concerning the rate of price inflation and the real output.[4]

We can rewrite equations (2.1) and (2.2) as follows.

$$\pi^e_{t+1} - \pi_t = \alpha(\bar{y} - y_t) - \varepsilon_t, \ (\alpha > 0) \tag{2.5}$$

$$y^e_{t+1} - y_t = \beta(r_t - \pi^e_{t+1} - \rho_o) - \xi_t, \ (\beta > 0). \tag{2.6}$$

Substituting (2.3) into these two equations, we have the following system of equations.

$$\pi^e_{t+1} - \pi_t = \alpha(\bar{y} - y_t) - \varepsilon_t = F_1(y_t; \varepsilon_t), \tag{2.7a}$$

$$\begin{aligned} y^e_{t+1} - y_t &= \beta\{(\gamma_1 - 1)(\pi_t - \bar{\pi}) + (\alpha + \gamma_2)(y_t - \bar{y}) + \varepsilon_t\} - \xi_t \\ &= F_2(\pi_t, y_t; \varepsilon_t, \xi_t), \end{aligned} \tag{2.7b}$$

$$\pi^e_{t+1} = E_t \pi_{t+1}, \qquad y^e_{t+1} = E_t y_{t+1}. \tag{2.7c}$$

3. Mathematical Analysis of the System without Exogenous Disturbance

The system (2.7) is called the DSGE (Dynamic Stochastic General Equilibrium) model if at least one of the exogenous disturbances ε_t and ξ_t is a stochastic variable. On the other hand, this system is called DGE (Dynamic General Equilibrium) model if there is no stochastic disturbance.[5] In this section, we shall consider the case without exogenous disturbance, that is to say,

$$\varepsilon_t = \xi_t = 0. \tag{3.1}$$

In this case, the *rational expectation* hypothesis, that is expressed by (2.7c) , is reduced to the following *perfect foresight* hypothesis.

$$\pi^e_{t+1} = \pi_{t+1}, \ y^e_{t+1} = y_{t+1}. \tag{3.2}$$

[2] In fact, these equations are log-linear approximations of the optimal conditions of the economic agents.

[3] See Taylor(1993) as for the original formulation of the Taylor rule.

[4] E_t stands for the operator of the mathematical expectation.

[5] See Woodford(2003), Bénassy(2007), Galí(2008), and Chiarella, Flaschel and Semmler(2013).

In such a case, equations (2.5) and (2.6) become

(3.3) $$\Delta\pi_t = \pi_{t+1} - \pi_t = \alpha(\bar{y} - y_t),$$

(3.4) $$\Delta y_t = y_{t+1} - y_t = \beta(r_t - \pi_{t+1} - \rho_o),$$

and the system (2.7) becomes as follows.

(3.5a) $$\pi_{t+1} = \pi_t + \alpha(\bar{y} - y_t) = G_1(\pi_t, y_t),$$

(3.5b) $$y_{t+1} = y_t + \beta\{(\gamma_1 - 1)(\pi_t - \bar{\pi}) + (\alpha + \gamma_2)(y_t - \bar{y})\} = G_2(\pi_t, y_t).$$

The equilibrium solution (π^*, y^*) of this system becomes simply

(3.6) $$\pi^* = \bar{\pi},\ y^* = \bar{y},$$

and the Jacobian matrix J_1 of this system becomes as follows.

(3.7) $$J_1 = \begin{pmatrix} 1 & -\alpha \\ \beta(\gamma_1 - 1) & 1 + \beta(\alpha + \gamma_2) \end{pmatrix}.$$

Its characteristic equation is given by

(3.8) $$\varphi_1(\lambda) \equiv |\lambda I - J_1| = \lambda^2 + a_1\lambda + a_2 = 0,$$

where

(3.9) $$a_1 = -\operatorname{tr} J_1 = -2 - \beta(\alpha + \gamma_2) < 0,$$

(3.10) $$a_2 = |J_1| = 1 + \beta(\alpha + \gamma_2) + \alpha\beta(\gamma_1 - 1) = 1 + \beta(\alpha\gamma_1 + \gamma_2) > 1.$$

Then, we have the following set of relationships.

(3.11a) $$A_1 \equiv 1 + a_1 + a_2 = \alpha\beta(\gamma_1 - 1),$$

(3.11b) $$A_2 \equiv 1 - a_2 = -\beta(\alpha\gamma_1 + \gamma_2) < 0,$$

(3.11c) $$A_3 \equiv 1 - a_1 + a_2 = 4 + \alpha\beta + 2\beta\gamma_2 + \alpha\beta\gamma_1 > 0.$$

The equilibrium point of the system of difference equations (3.5) is dynamically stable if and only if $|\lambda_j| < 1$, where $\lambda_j,\ j = 1, 2$, are two characteristic roots of (3.8). It is well known that such a stability condition is in fact satisfied *if and only if* the following set of inequalities are satisfied.[6]

(3.12) $$A_j > 0,\ j = 1, 2, 3.$$

In addition, the discriminant of the characteristic roots is given by

(3.13) $$D \equiv a_1^2 - 4a_2 = \beta\{\beta(\alpha + \gamma_2)^2 + 4\alpha(1 - \gamma_1)\}.$$

[6] See Gandolfo(2009, p.60). The set of inequalities (3.12) is the two-dimensional version of the so-called *Schur-Cohn stability conditions* of difference equations.

That is to say, the characteristic equation (3.8) has a pair of conjugate complex roots if and only if $D < 0$.

Now, we can obtain the following result.

LEMMA 3.1. (1) *At least one root of* (3.8) *has the absolute value that is greater than* 1 *for all* $\gamma_1 > 0$*. In other words, equilibrium of* (3.5) *is* ***unstable****.*
(2) *The characteristic equation* (3.8) *has a pair of conjugate complex roots, if and only if the inequality* $\gamma_1 > 1 + \dfrac{\beta(\alpha+\gamma_2)^2}{4\alpha}$ *is satisfied.*

Proof. (1) Inequality (3.11b) means that at least one of the Schur-Cohn stability conditions (3.12) is violated.

(2) It follows from (3.13) that we have $D < 0$ if and only if the inequality $\gamma_1 > 1 + \dfrac{\beta(\alpha+\gamma_2)^2}{4\alpha}$ is satisfied. □

The equilibrium point of the system (3.5) is called *totally unstable if and only if* both of two roots of (3.8) have the absolute values that are greater than 1.

In what follows, we write

$$\zeta = \frac{\beta(\alpha+\gamma_2)^2}{4\alpha}.$$

To study whether the system is totally unstable or not, it is convenient to consider the inverse matrix of J_1, that is,

$$K \equiv J_1^{-1} = \frac{1}{1+\beta(\alpha\gamma_1+\gamma_2)}\begin{pmatrix} 1+\beta(\alpha+\gamma_2) & \alpha \\ \beta(1-\gamma_1) & 1 \end{pmatrix}. \tag{3.14}$$

Then, let us consider the following characteristic equation.

$$\psi(\lambda) \equiv |\lambda I - K| = \lambda^2 + b_1\lambda + b_2 = 0, \tag{3.15}$$

where

$$b_1 = -\operatorname{tr} K = \frac{-2-\beta(\alpha+\gamma_2)}{1+\beta(\alpha\gamma_1+\gamma_2)} < 0. \tag{3.16}$$

$$b_2 = |K| = \frac{1}{1+\beta(\alpha\gamma_1+\gamma_2)} < 1. \tag{3.17}$$

Therefore, we obtain the following relationships.

$$B_1 \equiv 1 + b_1 + b_2 = \frac{\beta\alpha(\gamma_1-1)}{1+\beta(\alpha\gamma_1+\gamma_2)}, \tag{3.18a}$$

$$B_2 \equiv 1 - b_2 = \frac{\beta(\alpha\gamma_1+\gamma_2)}{1+\beta(\alpha\gamma_1+\gamma_2)} > 0, \tag{3.18b}$$

$$B_3 \equiv 1 - b_1 + b_2 = \frac{4+\beta\{\alpha(1+\gamma_1)+2\gamma_2\}}{1+\beta(\alpha\gamma_1+\gamma_2)} > 0. \tag{3.18c}$$

The following lemma is a direct consequence of the relationships (3.18).

LEMMA 3.2. *Absolute values of all of two roots of* (3.8) *are greater than* 1, *if and only if* $\gamma_1 > 1$. *In other words, the equilibrium point of* (3.5) *is totally unstable if and only if* $\gamma_1 > 1$.

Proof. It follows from the Schur-Cohn stability condition with respect to the system of difference equations that we have a set of inequalities with respect to (3.18)

$$B_j > 0, \ (j = 1, 2, 3) \tag{3.19}$$

if and only if $\gamma_1 > 1$. (*Cf.* Footnote 6). This means that *all* of the absolute values of the roots of the characteristic equation (3.15) are less than 1 *if and only if* $\gamma_1 > 1$. This result together with Theorem A.1 in Appendix A implies Lemma 3.2. □

Lemmas 3.1 and 3.2 entail the following proposition.

PROPOSITION 3.1. *Let* λ_1 *and* λ_2 *stand for two roots of* (3.8).
(1) *If* $0 < \gamma_1 < 1$, *then* λ_1 *and* λ_2 *are real with* $0 < \lambda_1 < 1 < \lambda_2$. *In this case, the equilibrium of* (3.5) *becomes a* ***saddle point***.
(2) *If* $\gamma_1 = 1$, *then* λ_1 *and* λ_2 *are real, with* $\lambda_1 = 1 < \lambda_2$.
(3) *If* $1 < \gamma_1 < 1 + \zeta$, *then* λ_1 *and* λ_2 *are real, with* $1 < \lambda_1 < \lambda_2$. *In this case, the equilibrium of* (3.5) *becomes* ***totally unstable***, *and any solution path that starts from points off equilibrium* ***diverges monotonically***.
(4) *If* $\gamma_1 = 1 + \zeta$, *then two roots are duplicated such that* $1 < \lambda_1 = \lambda_2$.
(5) *If* $1 + \zeta < \gamma_1$, *then the two roots are a pair of conjugate complex roots such that* $|\lambda_1| = |\lambda_2| = \sqrt{\theta^2 + \omega^2} > 1$, *where* $\lambda_1 = \theta + i\omega, \lambda_2 = \theta - i\omega$, $\omega \neq 0$ *and* $i = \sqrt{-1}$. *In this case, the equilibrium of* (3.5) *becomes* ***totally unstable***, *and any solution path that starts from points off equilibrium* ***diverges cyclically***.

As for proof, see Appendix B.

4. Some Critical Assessments

The prototype NK dynamic model that was taken up in the previous sections has several problematical features. In this section, we shall try to provide some critical assessments of such a model.

4.1. Anomalous dynamics with *sign reversals*. The most notorious problem of the prototype NK dynamic model is the phenomenon that is called the *sign reversal in New Keynesian Phillips curve*, pointed out by Mankiw (2001) clearly.[7]

[7] As for the related topics, see also Asada, Chiarella, Flaschel and Franke(2010), Flaschel and Schlicit(2006), Franke(2007) and Romer(2006).

It follows from (3.3) that $\Delta\pi_t$ becomes a *decreasing* function of y_t. That is, the rate of price inflation continues to *decrease* whenever the actual real output level is *greater than* the natural real output level. This contradicts the empirical fact of the most of countries. Mankiw(2001) writes as follows.

> "Although the New Keynesian Phillips curve has many virtues, it also has one striking vice: It is completely odds with the facts. In particular, it cannot come even close to explaining the dynamic effects of monetary policy on inflation and unemployment. This harsh conclusion shows up several places in the recent literature, but judging from the continued popularity of this model, I think it is fair to say that its fundamental inconsistency with the facts is not very appreciated." (Mankiw, 2001, p. C52)

It is worth noting that there is *another* difficulty of sign reversal in this model, which Mankiw(2001) does not refer to. That is, it follows from (3.4) that Δy_t becomes an *increasing* function of $r_t - \pi_{t+1}$. This means that the real output level continues to *increase* whenever the actual real interest rate is *greater than* its equilibrium level. It is evident that this fact also contradicts the empirical fact of the most of countries.

4.2. Problematical *jump variable* technique. In the traditional economic dynamic models, that conforms with the orthodox mathematical concept of stability, the initial values of the endogenous variables are historically given. In this setting, the equilibrium point is considered to be *stable* if and only if all of the characteristic roots are *stable roots*, and it is considered to be *unstable* if at least one characteristic root is an *unstable root*.[8] Lemma 3.1(1) means that the prototype NK model is accompanied by equilibrium that is *unstable in the mathematically orthodox sense*. The equilibrium of NK dynamic model will never be reached. This undermines the basis of the NK theory, however. Hence, the NK literature adopts the trick that makes the unstable system try to mimic a 'stable' system by using the so-called *jump variable* technique.

In the NK dynamic model, both of two endogenous variables π_t and y_t are considered to be jump variables or *not-predetermined variables*, the initial values of which are *freely chosen* by the economic agents. Furthermore, it is assumed that

[8] A stable (resp. unstable) root is the root which has the absolute value that is less (resp. greater) than 1 in case of the system of difference equations, and it is the root which has the negative (resp. positive) real part in case of the system of differential equations.

by the economic agents *only the initial conditions of the endogenous variables are chosen so as to ensure the convergence to the equilibrium point.*[9]

Hence, in place of the concept of stability and instability, NK literature introduces the following concept of the *determinacy* and *indeterminacy*.[10]

DEFINITION 4.1. (1) *Suppose that a set of the initial values of the jump variables, which ensures the convergence to the equilibrium point, is* ***uniquely*** *determined. Then, the system is called* ***determinate***.

(2) *Suppose that there are* ***multiple*** *sets of the initial values of the jump variables which ensure the convergence to the equilibrium point. Then, the system is called* ***indeterminate***.

PROPOSITION 4.1. (1) *If* $0 < \gamma_1 < 1$, *then the system* (3.5) *is indeterminate.*
(2) *If* $1 < \gamma_1$, *the system* (3.5) *is determinate.*

The meanings of Proposition 4.1 are as follows.

The general solution of the system of difference equations (3.5) becomes

$$\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} = \sum_{j=1}^{2} \begin{pmatrix} C_j \\ D_j \end{pmatrix} \lambda_j^t + \begin{pmatrix} \bar{\pi} \\ \bar{y} \end{pmatrix}, \tag{4.1}$$

where the constants C_j and D_j must satisfy the following relationships

$$\begin{pmatrix} \lambda_j - 1 & \alpha \\ -\beta(\gamma_1 - 1) & \lambda_j - \{1 + \beta(\alpha + \gamma_2)\} \end{pmatrix} \begin{pmatrix} C_j \\ D_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ (j = 1, 2). \tag{4.2}$$

(See Gandolfo, 2009, Ch.5). Substituting (4.2) into (4.1) we obtain

$$\begin{pmatrix} \pi_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\lambda_1}{\alpha} \end{pmatrix} C_1 \lambda_1^t + \begin{pmatrix} 1 \\ \frac{1-\lambda_2}{\alpha} \end{pmatrix} C_2 \lambda_2^t + \begin{pmatrix} \bar{\pi} \\ \bar{y} \end{pmatrix}, \tag{4.3}$$

where constants C_1 and C_2 are determined by a set of initial values in the following way.

$$\pi_o = C_1 + C_2 + \bar{\pi}, \tag{4.4a}$$

$$y_o = \frac{1-\lambda_1}{\alpha} C_1 + \frac{1-\lambda_2}{\alpha} C_2 + \bar{y}. \tag{4.4b}$$

First, suppose that $0 < \gamma_1 < 1$. In this case, we can consider that λ_1 and λ_2 are real values such that $0 < \lambda_1 < 1$ and $1 < \lambda_2$ without loss of generality in view of

[9] Usually, this assumption is rationalized on the ground that the *transversality condition*, that requires the convergence to the equilibrium point, must be satisfied in order to ensure the behavior of the representative agents to be *truly optimal*.

[10] See Woodford(2003), Bénassy(2007), and Galí(2008). See Blanchard and Kahn(1980) as for the sophisticated mathematical treatment that is related to this concept.

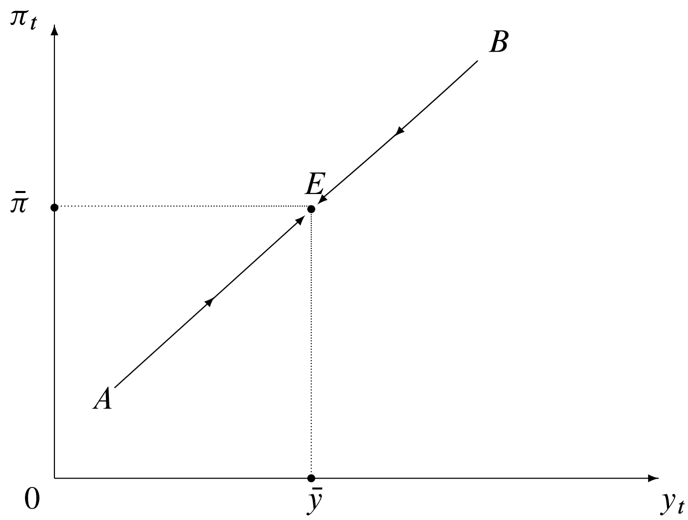

FIGURE 1. Case of $0 < \gamma_1 < 1$ (Indeterminate).

Proposition 3.1(1). In this situation, only the pair of the initial values (π_o, y_o) that satisfies the condition $C_2 = 0$ can ensure the convergence to the equilibrium point. This condition is reduced to the following set of equations in view of (4.4).

$$\pi_o = C_1 + \bar{\pi}, \tag{4.5a}$$

$$y_o = \frac{1-\lambda_1}{\alpha} C_1 + \bar{y}. \tag{4.5b}$$

Substituting (4.5b) into (4.5a), we obtain the following relationship between π_o and y_o that ensures the convergence to the equilibrium point.

$$\pi_o - \bar{\pi} = \frac{\alpha}{1-\lambda_1}(y_o - \bar{y}),\ 0 < \lambda_1 < 1. \tag{4.6}$$

Any pair (π_o, y_o) that is on the line in Figure 1 satisfies the relationship (4.6). Therefore, there are infinite numbers of the pairs of the initial values (π_o, y_o) which ensure the convergence to the equilibrium point.[11] In other words, the system becomes indeterminate in case of $0 < \gamma_1 < 1$.

Next, suppose that $\gamma_1 > 1$. In this case, both of the absolute values of λ_1 and λ_2 become greater than 1 (see Lemma 3.2). In this situation, only the pair of the initial values (π_o, y_o) that satisfies the condition $C_1 = C_2 = 0$ can ensure the convergence

[11] In fact, the line in Figure 1 is the *saddle path*.

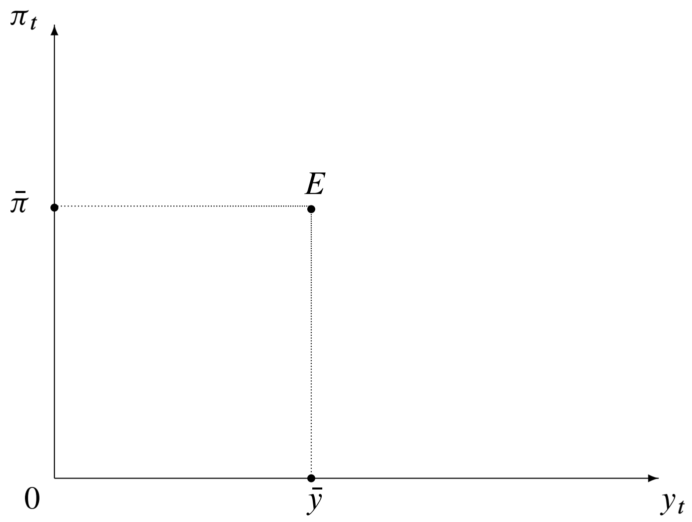

FIGURE 2. Case of $\gamma_1 > 1$ (Determinate).

to the equilibrium point. Substituting this condition into (4.4), we obtain

$$\pi_o = \bar{\pi}, \; y_o = \bar{y}. \tag{4.7}$$

This means that the system is *determinate* in case of $\gamma_1 > 1$, but in this case, the economic agents in this model select only the solution that is stuck to the equilibrium point E in Figure 2 for all times. In this case, *the economic fluctuation does not occur unless there are exogenous disturbances,* and *the dynamic model is reduced to the static model* virtually.

We think that *the adoption of such a jump variable technique in the NK dynamic model is unconvincing because of the following three reasons.*

Firstly, as Mankiw(2001) pointed out, actual economic data suggest that the rate of price inflation is not the jump variable that conveniently jumps discontinuously, but it is a *state variable* that moves sluggishly. Mankiw(2001) writes as follows.

> "In these models of staggered price adjustment, the price level adjusts slowly, but the inflation rate jumps quickly. Unfortunately for the model, that is not what we see in the data." (Mankiw, 2001, p. C54)

A proposal to resolve such a problem is to introduce the inertia into the model by considering the *backward-looking* factors. Typical method of reformulation is to replace (2.1) by the following new equation that includes the backward-looking factor π_{t-1} as well as the *forward-looking* factor π^e_{t+1}.[12]

(4.8) $$\pi_t = \theta\pi^e_{t+1} + (1-\theta)\pi_{t-1} + \alpha(y_t - \bar{y}) + \varepsilon_t,\ 0 < \theta < 1.$$

However, such a reformulation is rather *ad hoc* and it is not clear how it can be derived from the analytical framework of the NK dynamic model consistently.

Secondly, as we already pointed out, the prototype NK dynamic model cannot produce the economic fluctuations unless we introduce the exogenous disturbances into the model. This conclusion is not confined to the case of *determinacy* (the case of $\gamma_1 > 1$). It also applies to the case of *indeterminacy* (the case of $0 < \gamma_1 < 1$), because even in case of indeterminacy, the solution converges to the equilibrium point monotonically (see Figures 1 and 2). In other words, the endogenous economic fluctuation is impossible in this model.

Thirdly, some economists criticize the methodology of the representative agent approach of the NK dynamic model that is based on the analysis of the behavior of the omniscient and homogeneous representative agent. For example, Kirman(1992), Asada, Chiarella, Flaschel and Franke(2010) and Chiarella, Flaschel and Semmler(2013) proposed the approach that considers the interactions of the heterogeneous agents who behave bounded-rationally.[13] Let us quote from their papers.

> "A tentative conclusion, at this point, would be that the representative agent approach is fatally flawed because it attempts to impose order on the economy through the concept of an omniscient individual. In reality, individuals operate in very small subsets of the economy and interact with those with whom they have dealings. It may well be that out of this local but interacting activity emerges some sort of self organization which provides regularity at the macroeconomic level. ······ The equilibria of the worlds described by any of these approaches may be conceptually very different from those implied by the artifact of the representative individual. Cycles and fluctuations emerge not as the result of some substantial exogenous shock and the reaction to it of one individual, but as a natural result of interaction, together with occasional small changes or 'mutations' in the behavior of some individuals." (Kirman, 1992, pp.132-3)

[12] See, for example, Furler(1997). Romer(2006, Ch.6) and Franke(2007) contain the useful interpretations of this topic.

[13] Such an approach is somewhat similar to the approach of Akerlof and Shillar(2009) that is influenced by the discourse of Keynes(1936).

> “While the *microfoundation* of economic behavior is per se an important desideratum to be reflected also by behaviorally oriented macrodynamics, the use of ‘representative’ consumers and firms for the explanation of macroeconomic phenomena is too simplistic and also too narrow for a proper treatment of what is *really* interesting on the economic behavior of economic agents – the interaction of heterogeneous agents –, and it is also not detailed enough to discuss the various feedback channels present in the real world. $\cdots\cdots$ Indeed, agents are heterogeneous, form heterogeneous expectations along other lines than suggested by the rational expectations theory, and have different short and long term views about the economy.” (Asada, Chiarella, Flaschel and Franke, 2010, pp.205-6)

> “Central to our approach is the use of non-market clearing adjustment process that are indeed the basis for the consideration of the various Keynesian macroeconomic feedback effect, the Pigou real balance effect, the Tobin price expectation effect (where an expected price fall will trigger instabilities), and the Keynes effect in the context of a dynamic multiplier model. These have all been characteristic in the traditional Keynesian approaches to macrodynamics but have mostly been forgotten in modern macroeconomics.” (Chiarella, Flaschel and Semmler, 2013, p.110)

5. Mathematical Analysis of the Effects of the Exogenous Disturbance in a Prototype New Keynesian Dynamic Model

In this section, we introduce the exogenous disturbance into the model and study its effects mathematically.[14] In (2.7), we assume that $\varepsilon_t = 0$ and ξ_t are exogenous but non-stochastic disturbances, that is,

(5.1a) $$\pi_{t+1} = \pi_t + \alpha(\bar{y} - y_t),$$

(5.1b) $$y_{t+1} = y_t + \beta\{(\gamma_1 - 1)(\pi_t - \bar{\pi}) + (\alpha + \gamma_2)(y_t - \bar{y})\} - \xi_t.$$

Furthermore, let us assume as follows.

(5.2) $$\text{For all } t \in \{0, 1, 2, \ldots, t_o - 1\},\ \pi_t = \bar{\pi},\ y_t = \bar{y}.$$

[14] As we see in this section, the existence of the exogenous disturbance is essential for the occurrence of the economic fluctuations in the prototype NK dynamic model, but the stochastic element does not play the essential role.

$$\xi_t = \begin{cases} 0 & \text{for } t \in \{0, 1, 2, \dots, t_o - 1\}, \\ \xi_{t_o} \neq 0 & \text{for } t = t_o, \\ \xi_{t_o}\kappa^{(t-t_o)} & \text{for } t \in \{t_o + 1, t_o + 2, \dots\}, \end{cases} \tag{5.3}$$

where ξ_{t_o} is given, and $0 < \kappa < 1$.

Then, we have the following three-dimensional system of difference equations for all time periods $t \in \{t_o, t_o + 1, \dots\}$:

$$\pi_{t+1} = \pi_t + \alpha(\bar{y} - y_t) = G_1(\pi_t, y_t), \tag{5.4a}$$

$$\begin{aligned} y_{t+1} &= y_t + \beta\{(\gamma_1 - 1)(\pi_t - \bar{\pi}) + (\alpha + \gamma_2)(y_t - \bar{y})\} - \xi_t \\ &= G_2(\pi_t, y_t, \xi_t), \end{aligned} \tag{5.4b}$$

$$\xi_{t+1} = \kappa\xi_t. \tag{5.4c}$$

The Jacobian matrix of this system becomes

$$J_2 = \begin{pmatrix} 1 & -\alpha & 0 \\ \beta(\gamma_1 - 1) & 1 + \beta(\alpha + \gamma_2) & -1 \\ 0 & 0 & \kappa \end{pmatrix}, \tag{5.5}$$

and the characteristic equation of this system is given by

$$\varphi_2(\lambda) \equiv |\lambda I - J_2| = |\lambda I - J_1|(\lambda - \kappa) = 0, \tag{5.6}$$

where J_1 is defined by (3.7).

This characteristic equation has a real root $\lambda_3 = \kappa$ and other two roots are determined by (3.8). To simplify the analysis, we consider only the case of

$$1 < \gamma_1 < 1 + \zeta. \tag{5.7}$$

It follows from Proposition 3.1(3) that (3.8) has two real roots such that $1 < \lambda_1 < \lambda_2$. In this case, the general solution of (5.4) for $t \geq t_o$ becomes

$$\begin{pmatrix} \pi_t \\ y_t \\ \xi_t \end{pmatrix} = \sum_{j=1}^{3} \begin{pmatrix} C_j \\ D_j \\ E_j \end{pmatrix} \lambda_j^t + \begin{pmatrix} \bar{\pi} \\ \bar{y} \\ 0 \end{pmatrix}, \tag{5.8}$$

where $1 < \lambda_1 < \lambda_2$, $0 < \kappa < 1$, and the relationships that, for $j = 1, 2, 3$,

$$\begin{pmatrix} \lambda_j - 1 & \alpha & 0 \\ -\beta(\gamma_1 - 1) & \lambda_j - \{1 + \beta(\alpha + \gamma_2)\} & 1 \\ 0 & 0 & \lambda_j - \kappa \end{pmatrix} \begin{pmatrix} C_j \\ D_j \\ D_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.9}$$

must be satisfied (see Gandolfo, 2009, Ch.9).

Rewriting (5.8) by using (5.9), we obtain the following relationship.

$$(5.10)\quad \begin{pmatrix}\pi_t\\ y_t\\ \xi_t\end{pmatrix} = \begin{pmatrix}1\\ \frac{1-\lambda_1}{\alpha}\\ 0\end{pmatrix} C_1\lambda_1^{t-t_o} + \begin{pmatrix}1\\ \frac{1-\lambda_2}{\alpha}\\ 0\end{pmatrix} C_2\lambda_2^{t-t_o} + \begin{pmatrix}\frac{\alpha}{\tau}\\ \frac{1-\kappa}{\tau}\\ 1\end{pmatrix} E_3\kappa^{t-t_o} + \begin{pmatrix}\bar{\pi}\\ \bar{y}\\ 0\end{pmatrix},$$

where

$$\tau = \alpha\beta(\gamma_1 - 1) + (1-\kappa)\{1 + \beta(\alpha + \gamma_2) - \kappa\} > 0,$$

and C_1, C_2 and E_3 are the constants which are determined by the initial values $(\pi_{t_o}, y_{t_o}, \xi_{t_o})$. In fact, we have the following relationships from (5.10).

$$(5.11a)\quad \pi_{t_o} = C_1 + C_2 + \frac{\alpha}{\tau}\xi_{t_o} + \bar{\pi},$$

$$(5.11b)\quad y_{t_o} = \frac{1-\lambda_1}{\alpha}C_1 + \frac{1-\lambda_2}{\alpha}C_2 + \frac{1-\kappa}{\tau}\xi_{t_o} + \bar{y},$$

$$(5.11c)\quad \xi_{t_o} = E_3.$$

We can determine the values of C_1, C_2 and E_3 by solving (5.11) if a triplet of the initial values $(\pi_{t_o}, y_{t_o}, \xi_{t_o})$ is predetermined. However, in the NK setting, the initial values π_{t_o} and y_{t_o} are considered to be *not-predetermined variables* which can be *chosen freely* by the economic agents, although the initial value of the exogenous disturbance (ξ_{t_o}) is a predetermined variable.

Furthermore, it is assumed that the economic agents *choose* only the pair of the initial values (π_{t_o}, y_{t_o}) which ensures that the solution (5.10) converges to the equilibrium point $(\bar{\pi}, \bar{y})$. It is apparent that only the initial values that satisfy the condition

$$(5.12)\quad C_1 = C_2 = 0$$

can ensure the convergence to the equilibrium point. Substituting (5.12) into (5.11), we obtain the following requirement.

$$(5.13a)\quad \pi_{t_o} = \frac{\alpha}{\tau}\xi_{t_o} + \bar{\pi},$$

$$(5.13b)\quad y_{t_o} = \frac{1-\kappa}{\tau}\xi_{t_o} + \bar{y}.$$

From the assumptions $\gamma_1 > 1$, $0 < \kappa < 1$ and (5.13), we have the relationships

$$(5.14)\quad \mathrm{sign}(\pi_{t_o} - \bar{\pi}) = \mathrm{sign}(y_{t_o} - \bar{y}) = \mathrm{sign}\,\xi_{t_o},$$

and the special solution that ensures the convergence to the equilibrium point becomes as follows.

(5.15a) $$\pi_t = \frac{\alpha}{\tau}\xi_{t_o}\kappa^{t-t_o} + \bar{\pi},$$

(5.15b) $$y_t = \frac{1-\kappa}{\tau}\xi_{t_o}\kappa^{t-t_o} + \bar{y},$$

(5.15c) $$\xi_t = \xi_{t_o}\kappa^{t-t_o}.$$

Figure 3 is the so-called *impulse response functions* that are produced by the special solution (5.15) in case of $\xi_{t_o} < 0$. In this figure, it is supposed that the system is at the equilibrium point for $0 \leq t < t_o$, the exogenous disturbance is caused at the time period $t = t_o$, and then it diminishes continuously. Responding to this exogenous disturbance, the pair of the values of π and y suddenly *jumps* to the initial condition (π_{t_o}, y_{t_o}) that ensures the convergence to the equilibrium point conveniently. *The jump variable postulate manages to conceal the paradoxical behavior of the main variables.*

However, it is not clear why the economic agents in this model can choose the special initial values that are given by (5.13) immediately after the *exogenous shock* occurs. What occurs if another combination of the initial conditions is chosen *by mistake*?

As a thought experiment, let us assume that

(5.16) $$\pi_{t_o} = \bar{\pi},\ y_{t_o} = \bar{y},$$

even if $\xi_{t_o} \neq 0$, which means that the economic variables cannot conveniently jump even if an exogenous shock occurs.

Substituting (5.16) into (5.11), we have the following system of equations.

(5.17) $$\begin{pmatrix} 1 & 1 \\ \frac{1-\lambda_1}{\alpha} & \frac{1-\lambda_2}{\alpha} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = -\frac{\xi_{t_o}}{\tau} \begin{pmatrix} \alpha \\ 1-\kappa \end{pmatrix}.$$

Solving this equation with respect to C_1 and C_2, we obtain the following result.

(5.18a) $$C_1 = \frac{-\alpha(\lambda_2-\kappa)}{(\lambda_2-\lambda_1)\tau}\xi_{t_o},$$

(5.18b) $$C_2 = \frac{\alpha(\lambda_1-\kappa)}{(\lambda_2-\lambda_1)\tau}\xi_{t_o}.$$

Substituting (5.18) and $E_3 = \xi_{t_o}$ into (5.10), we obtain the special solution of the system when (5.16) is assumed.

Since it is assumed that $0 < \kappa < 1 < \lambda_1 < \lambda_2$, (5.18) implies that

(5.19) $$-\operatorname{sign} C_1 = \operatorname{sign} C_2 = \operatorname{sign} \xi_{t_o}.$$

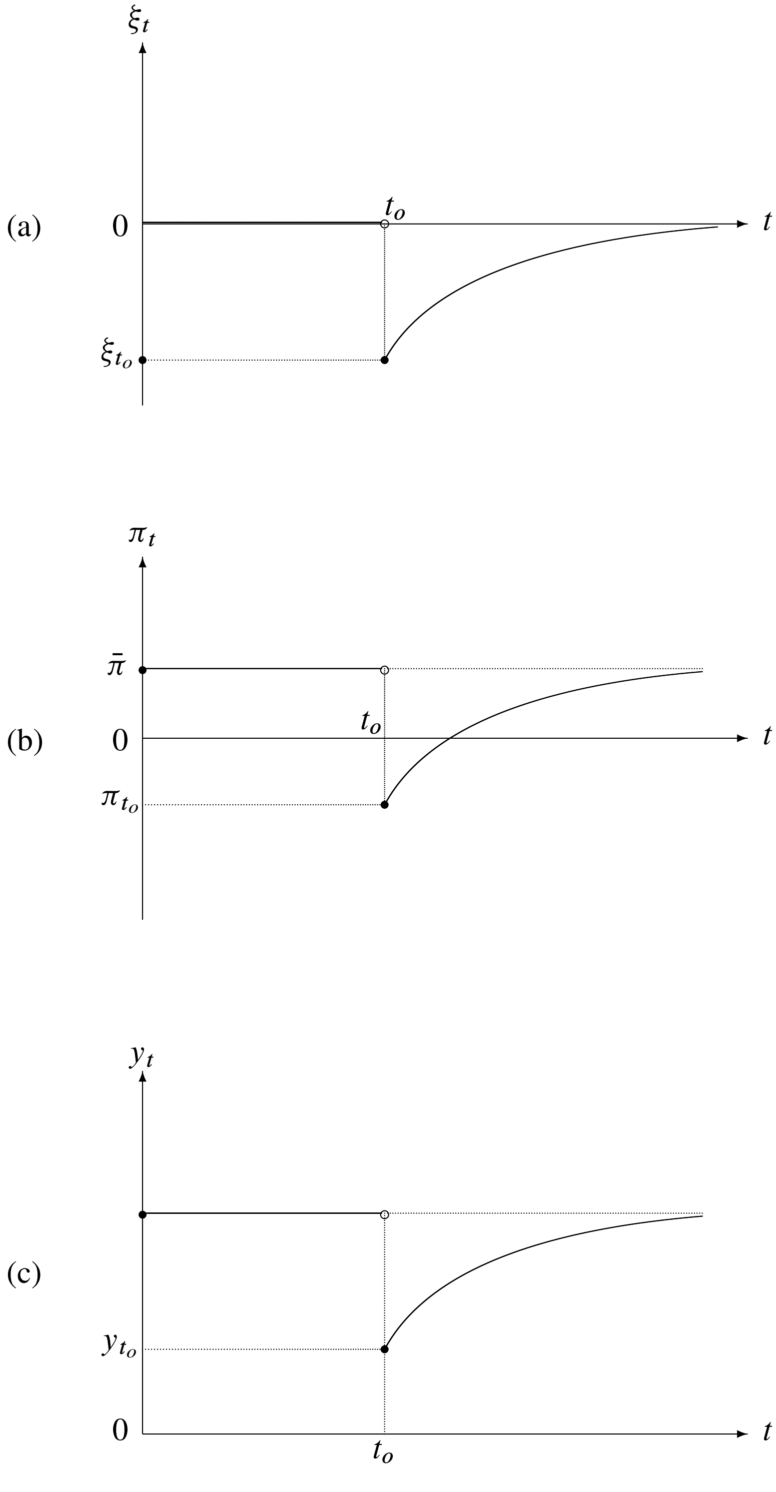

FIGURE 3. The special solution (5.15) in case of $\xi_{t_o} < 0$.

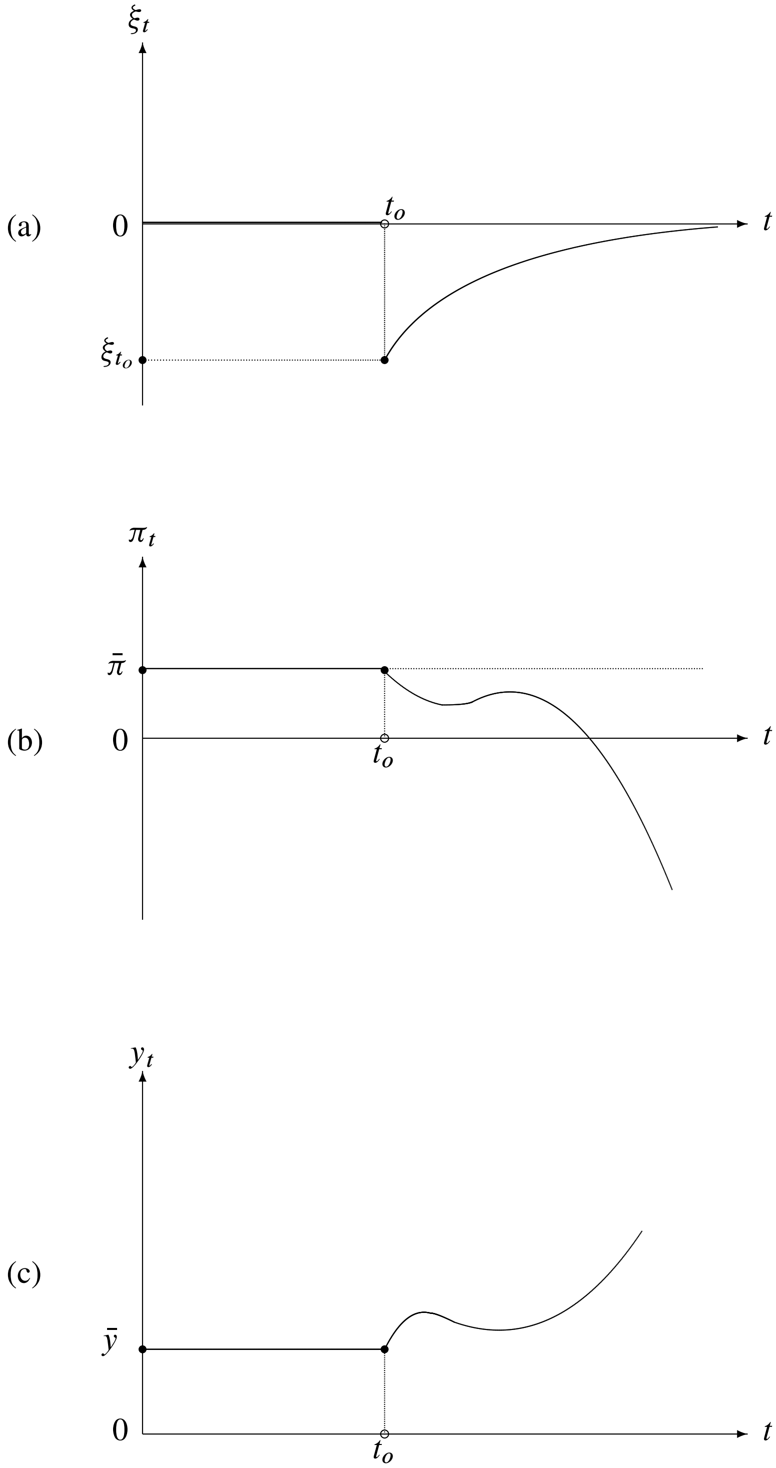

FIGURE 4. The special solution (5.10) in case of $\pi_{t_o} = \bar{\pi}$, $y_{t_o} = \bar{y}$, $\xi_{t_o} < 0$.

In this case, the propagation process of the effects of the exogenous disturbance becomes as follows. For simplicity, we consider only the case of $\xi_{t_o} < 0$.

If (5.16) is satisfied, we have the following from (5.4) in case of $\xi_{t_o} < 0$.

(5.20a) $\pi_{t_o+1} = \bar{\pi} = \pi_{t_o}$,

(5.20b) $y_{t_o+1} = \bar{y} - \xi_{t_o} > \bar{y} = y_{t_o}$,

(5.20c) $\pi_{t_o+2} = \bar{\pi} + \alpha(\bar{y} - y_{t_o+1}) = \bar{\pi} + \alpha\xi_{t_o} < \bar{\pi} = \pi_{t_o+1}$,

(5.20d) $y_{t_o+2} = y_{t_o+1} + (\alpha + \gamma_2)(y_{t_o+1} - \bar{y}) = y_{t_o+1} - (\alpha + \gamma_2)\xi_{t_o} > y_{t_o+1}$.

A set of relationships (5.20) means that the *fall* of π and the *rise* of y are caused soon after this exogenous shock. This tendency is not restricted to the periods soon after the shock.

We have $C_1 > 0$, $C_2 < 0$ and $1 < \lambda_1 < \lambda_2$ where λ_2 is the dominant root that governs the dynamic behavior of the system in case of $C_2 \neq 0$. Therefore, (5.10) implies that π continues to *fall* and y continues to *rise* ultimately, which means that the *deflationary prosperity* rather than the *deflationary depression* is caused by this exogenous shock. Figure 4 illustrates the typical impulse response functions of such unconvincing and paradoxical behaviors of the main variables that are caused by the exogenous disturbance.

6. Toward an Alternative *Old Keynesian* Approach

Up to the previous section, we presented an analytical critique of the prototype NK dynamic model. The point is that the prototype NK dynamic model produces the anomalous behaviors that are inconsistent with the empirical facts.

In this section, we develop a constructive and alternative approach that was presented by Asada(2012), which is called the *Old Keynesian* dynamic model after Tobin(1969, 1994).[15]

Asada's(2012) Old Keynesian dynamic model consists of the following set of equations.[16]

(6.1) $\pi_t = x_t + f(y_t);\ f(\bar{y}) = 0,\ f_y = f'(y_t) > 0,$

(6.2) $y_t = z_t + g(r_t - x_t);\ g_{r-x} = g'(r_t - x_t) < 0,$

(6.3) $x_t = \pi^e_{t+1},\ z_t = y^e_{t+1},$

[15] The work, such as Asada, Chiarella, Flaschel and Franke(2003, 2010), Chiarella, Flaschel and Franke(2005), and Asada, Flaschel, Mouakil and Proanõ(2011), is also based on the Old Keynesian tradition of the modeling methodology.

[16] In this formulation, the exogenous disturbance is neglected for simplicity.

$$\Delta r_t = r_{t+1} - r_t = \begin{cases} \gamma_1(\pi_t - \bar{\pi}) + \gamma_2(y_t - \bar{y}), & \text{if } r_t > 0, \\ \max[0, \gamma_1(\pi_t - \bar{\pi}) + \gamma_2(y_t - \bar{y})], & \text{if } r_t = 0; \end{cases} \tag{6.4}$$

$$\Delta x_t = x_{t+1} - x_t = \delta[\theta_1(\bar{\pi} - x_t) + (1-\theta_1)(\pi_t - x_t)],\ 0 \leq \theta_1 \leq 1, \tag{6.5}$$

$$\Delta z_t = z_{t+1} - z_t = \sigma[\theta_2(\bar{y} - z_t) + (1-\theta_2)(y_t - z_t)],\ 0 \leq \theta_2 \leq 1, \tag{6.6}$$

where x_t stands for the expected rate of price inflation, z_t the expected real national income, and $r_t - x_t$ the expected real interest rate; γ_1, γ_2, δ and σ are positive parameters. Meanings of other symbols are the same as those in the previous sections;

Equations (6.1) and (6.2) are the *expectation-augmented Phillips curve* and the *expectation-augmented IS curve* respectively. In these formulations, it is allowed for that the functions $f(y_t)$ and $g(r_t - x_t)$ are nonlinear functions.[17] (6.4) is a Taylor-rule-type interest rate monetary policy rule. In this formulation, the natural nonlinearity that is due to the nonnegative constraint of the nominal interest rate is explicitly considered. Equations (6.5) and (6.6) are the expectation formation equations concerning the rate of price inflation and the real national income. They are the mixtures of the forward-looking and backward-looking (adaptive) expectation formations, although they are different from the NK rational expectation formations. We can consider that θ_1 is the parameter that reflects the *credibility* of the central bank's inflation targeting, and θ_2 is the parameter that reflects the credibility of the central bank's employment targeting. We can consider that the formulations (6.5) and (6.6) are the simplest versions of the *heterogeneous expectation hypothesis*.

Incidentally, we have the following equations by substituting (6.1) and (6.2) into (6.5) and (6.6).

$$\Delta x_t = \delta[\theta_1(\bar{\pi} - x_t) + (1-\theta_1)f(y_t))],\ f'(y_t) > 0, \tag{6.7}$$

$$\Delta z_t = \sigma[\theta_2(\bar{y} - z_t) + (1-\theta_2)g(r_t - x_t)],\ g'(r_t - x_t) < 0. \tag{6.8}$$

These equations imply that the *rate of change* of the expected rate of inflation Δx_t is an *increasing* function of the actual real national income (y_t) and the *rate of change* of the expected real national income Δz_t is a *decreasing* function of the expected real interest rate $(r_t - x_t)$. In other words, this *Old Keynesian dynamic model is immune from the notorious NK sign reversal problems.*

Asada(2012) showed that this system can be reduced to the following three dimensional system of nonlinear difference equations. First, put

$$\mu = \gamma_1\{x_t + f(z_t + g(r_t - x_t)) - \bar{\pi}\} + \gamma_2\{z_t + g(r_t - x_t) - \bar{y}\},$$

[17] The empirical data of the Japanese economy suggests that $f_{yy} = f''(y_t) > 0$.

and we have the following.

(6.9a) $$\Delta r_t = r_{t+1} - r_t = H_1(r_t, x_t, z_t; \gamma_1, \gamma_2) = \begin{cases} \mu, & \text{if } r_t > 0, \\ \max[0, \mu], & \text{if } r_t = 0; \end{cases}$$

(6.9b) $$\Delta x_t = x_{t+1} - x_t = H_2(r_t, x_t, z_t; \delta, \theta_1)$$
$$= \delta[\theta_1(\bar{\pi} - x_t) + (1 - \theta_1) f(z_t + g(r_t - x_t))],\ 0 \leq \theta_1 \leq 1,$$

(6.9c) $$\Delta z_t = z_{t+1} - z_t = H_3(r_t, x_t, z_t; \sigma, \theta_2)$$
$$= \sigma[\theta_2(\bar{y} - z_t) + (1 - \theta_2) g(r_t - x_t)],\ 0 \leq \theta_2 \leq 1.$$

Asada(2012) studied the following differential equation version of (6.9).

(6.10a) $$\dot{r} = H_1(r, x, z; \gamma_1, \gamma_2),$$

(6.10b) $$\dot{x} = H_2(r, x, z; \delta, \theta_1),$$

(6.10c) $$\dot{z} = H_3(r, x, z; \sigma, \theta_2).$$

Now, we shall outline Asada's(2012) analysis. The equilibrium solution of this system that ensures $\dot{r} = \dot{x} = \dot{z} = 0$ can be written as follows, if we ignore the nonnegative constraint of r.

(6.11) $$r^* = \rho_o + \bar{\pi},\ x^* = \pi^* = \bar{\pi},\ z^* = y^* = \bar{y}.$$

It is clear that we have $r^* > 0$ *if and only if* the inequality

(6.12) $$\bar{\pi} > -\rho_o$$

is satisfied.

In this case, the equilibrium real interest rate becomes

(6.13) $$r^* - \pi^* = \rho_o.$$

We assume that the inequality (6.12) is in fact satisfied.[18] Then, the Jacobian matrix of this system *at the equilibrium point* becomes

(6.14) $$J_3 = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where

(6.15) $$H_{11} = (\gamma_1 f_y + \gamma_2) g_{r-x} < 0,\quad H_{12} = \gamma_1(1 - f_y g_{r-x}) - \gamma_2 g_{r-x} > 0,$$
$$H_{13} = \gamma_1 f_y + \gamma_2 > 0,\qquad H_{21} = \delta(1 - \theta_1) f_y g_{r-x} \leq 0,$$
$$H_{22} = -\delta[\theta_1 + (1 - \theta_1) f_y g_{r-x}],\qquad H_{23} = \delta(1 - \theta_1) f_y \geq 0,$$

[18] This inequality is automatically satisfied if $\rho_o > 0$ and $\bar{\pi} > 0$.

$$H_{31} = \sigma(1-\theta_2)g_{r-x} \leq 0, \quad H_{32} = -\alpha\sigma(1-\theta_2)g_{r-x} \geq 0,$$
$$H_{33} = \sigma\theta_2 \leq 0.$$

The characteristic equation of this system at the equilibrium point becomes as follows.

(6.16) $$\varphi_3(\lambda) \equiv |\lambda I - J_3| = \lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 = 0,$$

where

(6.17a) $$d_1 = -\operatorname{tr} J_3 = -H_{11} - H_{22} - H_{33},$$

(6.17b) $$d_2 = \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} + \begin{vmatrix} H_{11} & H_{13} \\ H_{31} & H_{33} \end{vmatrix} + \begin{vmatrix} H_{22} & H_{23} \\ H_{32} & H_{33} \end{vmatrix},$$

(6.17c) $$d_3 = -|J_3| > 0, \quad \text{if } 0 \leq \theta_1 < 1 \text{ and } 0 \leq \theta_2 < 1.$$

In this Old Keynesian model, the traditional concept of (in)stability is adopted. In other words, the jump variables are not allowed for in this model, and the equilibrium point is considered to be *locally stable if and only if* all of the roots of the characteristic equation (6.16) have negative real parts.[19] It is well known that such local stability condition is satisfied *if and only if* the following *Routh-Hurwitz conditions* are satisfied (see Gandolfo, 2009, Ch.16).

(6.18) $$d_j > 0 \quad (j = 1, 2, 3), \text{ and } \quad d_1 d_2 - d_3 > 0.$$

By utilizing this result, Asada(2012) proved the following two propositions.

PROPOSITION 6.1. *Suppose that the parameters γ_1 and γ_2 are fixed at any positive values. Then, the equilibrium point of* (6.10) *is* ***unstable*** *if either of the following two conditions* (a) *or* (b) *is satisfied.*

(a) *$\delta > 0$ is sufficiently large and θ_1 is sufficiently small* (*close to zero*).

(b) *The condition $0 \leq \theta_1 < 1$ is satisfied, $\delta > 0$ and $\sigma > 0$ are sufficiently large, and θ_2 is sufficiently small* (*close to zero*).

PROPOSITION 6.2. (1) *Suppose that the parameters γ_1, γ_2, δ and σ are fixed at any positive values. The equilibrium point of* (6.10) *is* ***locally asymptotically stable****, if both of θ_1 and θ_2 are close to* 1 (*including the cases of $\theta_1 = 1$ and $\theta_2 = 1$*).
(2) *Suppose that the parameter values δ, σ, θ_1 and θ_2 are fixed at any values such that $\delta > 0$, $\sigma > 0$, $0 \leq \theta_1 < 1$, and $0 \leq \theta_2 < 1$. Then, the equilibrium point of* (6.10) *is* ***locally asymptotically stable****, if either of $\gamma_1 > 0$ or $\gamma_2 > 0$ is sufficiently large.*

[19] The same traditional concept of stability/instability is adopted also in *Post Keynesian* literature, such as Keen(2000), Asada(2001), and Charles(2008).

These two propositions show that the increases (resp. decreases) of the parameter values θ_1, θ_2, γ_1 and γ_2 have the *stabilizing* (resp. *destabilizing*) effects in the traditional sense in this Old Keynesian model. In other words, the equilibrium point of the dynamic system (6.10) is stable (resp. unstable) if the credibility of the inflation targeting and the employment targeting by the central bank is high (resp. low) and the central bank's monetary policy is active (resp. inactive).

Let us select one of such parameters, for example, θ_1 as a *bifurcation parameter*, and suppose that the equilibrium point of the system is *unstable* for all sufficiently small values of θ_1, and it is *locally stable* for all values of θ_1 that are sufficiently close to 1. Then, there exists at least one *bifurcation point* $\theta_1^o \in (0, 1)$ at which the discontinuous switch from the unstable system to the stable system occurs as the parameter value θ_1 increases.

It is apparent that at least one real part of the roots of the characteristic equation (6.16) must be zero at the bifurcation point. However, we cannot have the real root $\lambda = 0$ at such a bifurcation point, because we have

$$\text{(6.19)} \qquad \varphi_3(0) = d_3 \neq 0, \qquad \text{if } 0 \leq \theta_1 < 1 \text{ and } 0 \leq \theta_2 < 1$$

from equations (6.16) and (6.17c). This means that the characteristic equation (6.16) has a pair of pure imaginary roots at the bifurcation point. This situation is enough to apply the *Hopf bifurcation theorem* (Appendix C) to the dynamic system (6.10), and we have the following proposition.

PROPOSITION 6.3. *The bifurcation point of the dynamic system* (6.10)*, which is the point at which the discontinuous switch from the unstable system to the stable system occurs as one of the parameter values* θ_1, θ_2, γ_1 *and* γ_2 *increases, is in fact the Hopf bifurcation point. In other words, the non-constant closed orbits exist at some range of the parameter values that are sufficiently close to the bifurcation point.*

Proposition 6.3 shows that the *endogenous cyclical fluctuations* occur at some range of intermediate values of the parameters θ_1, θ_2, γ_1 and γ_2, even if there is no exogenous disturbance, contrary to the prototype NK dynamic model that cannot produce the economic fluctuations without the exogenous disturbance.

7. Concluding Remarks

In this paper, we showed that the prototype *New Keynesian*(NK) dynamic model produces several paradoxical and anomalous behaviors, which are inconsistent with the empirical facts. *Sign reversals* of NK Philips curve and NK IS curve are typical examples of such anomaly. We also observed that the problematical trick of the *jump variable* technique is an unconvincing device. Finally, we presented the outline of the alternative *Old Keynesian* dynamic model that is consistent with the empirical facts and immune from the NK anomalies.

Appendix A. The Total Instability Theorem

The following purely mathematical result is useful for the proof of Lemma 3.2 in the text.

THEOREM A.1. *Let A be an $n \times n$ matrix with the property $|A| \neq 0$. Then, the absolute values of all roots of the characteristic equation $|\lambda I - A| = 0$ are greater than 1 if and only if the absolute values of all roots of the characteristic equation $|\lambda I - A^{-1}| = 0$ are less than 1.*

Proof. The assumption $|A| \neq 0$ means that 0 cannot be a characteristic root of the equation $|\lambda I - A| = 0$. Let $\lambda_A \neq 0$ be a characteristic root of this equation that satisfies $|\lambda_A I - A| = 0$. Then, $\lambda I - A = (\lambda A^{-1} - I)A = \lambda I(A^{-1} - \lambda^{-1} I)A$, so that $|\lambda I - A| = (-1)^n \lambda^n |\lambda^{-1} I - A^{-1}||A|$. Hence, $|\lambda I - A| = 0$ if and only if $|\lambda^{-1} I - A^{-1}| = 0$.

This means that $\lambda_B \equiv \frac{1}{\lambda_A}$ is a characteristic root of the equation $|\lambda I - B| = 0$, where $B \equiv A^{-1}$.

Suppose that λ_A is a real number. In this case, it is obvious that $|\lambda_A| > 1$ *if and only if* $|\lambda_B| < 1$.

Next, suppose that λ_A is a complex number such that $\lambda_A = \theta + i\omega$, $i = \sqrt{-1}$, $\omega \neq 0$. Then, we have $\lambda_B = \dfrac{1}{\theta + i\omega} = \dfrac{\theta - i\omega}{\theta^2 + \omega^2}$. In this case, we have $|\lambda_A| = \sqrt{\theta^2 + \omega^2}$ and $|\lambda_B| = \dfrac{1}{\sqrt{\theta^2 + \omega^2}}$. This means that $|\lambda_A| > 1$ *if and only if* $|\lambda_B| < 1$ even if λ_A is a complex number. □

Appendix B. Proof of Proposition 3.1.

The characteristic equation (3.8) has the following properties.

(B.1) $\varphi_1'(\lambda) = 2\lambda + a_1$,
(B.2) $\varphi_1(0) = a_2 > 0$,
(B.3) $\varphi_1'(0) = a_1 < 0$,
(B.4) $\varphi_1(1) = 1 + a_1 + a_2 = \alpha\beta(\gamma_1 - 1)$,
(B.5) $\varphi_1'(1) = 2 + a_1 = -\beta(\alpha + \gamma_2) < 0$.

These properties entail that the following statements hold.

(1) Suppose that $0 < \gamma_1 < 1$. Then, we have $\varphi_1(0) > 0$, $\varphi_1'(0) < 0$, $\varphi_1(1) < 0$ and $\varphi_1'(1) < 0$. In this case, we have Figure 5, which implies that we have two real roots, λ_1 and λ_2, such that $0 < \lambda_1 < 1 < \lambda_2$.

(2) Suppose that $\gamma_1 = 1$. Then, we have $\varphi_1(0) > 0$, $\varphi_1'(0) < 0$, $\varphi_1(1) = 0$ and $\varphi_1'(1) < 0$. In this case, we have Figure 6, which implies that we have two real roots, λ_1 and λ_2, such that $\lambda_1 = 1 < \lambda_2$.

(3) Suppose that $1 < \gamma_1 < 1 + \zeta$. Then, we have $\varphi_1(0) > 0$, $\varphi_1'(0) < 0$, $\varphi_1(1) > 0$ and $\varphi_1'(1) < 0$, with $D > 0$. In this case, we have Figure 7, which implies that we have two real roots, λ_1 and λ_2, such that $1 < \lambda_1 < \lambda_2$.

(4) Suppose that $\gamma_1 = 1 + \zeta$. Then, we have $\varphi_1(0) > 0$, $\varphi_1'(0) < 0$, $\varphi_1(1) > 0$ and $\varphi_1'(1) < 0$, with $D = 0$. In this case, we have Figure 8, which implies that we have a pair of multiple roots such that $1 < \lambda_1 = \lambda_2$.

(5) Suppose that $1 + \zeta < \gamma_1$. Then, it follows from Lemma 3.2 and $D < 0$ that the characteristic equation (3.8) has a pair of conjugate complex roots and their absolute values are greater than 1.

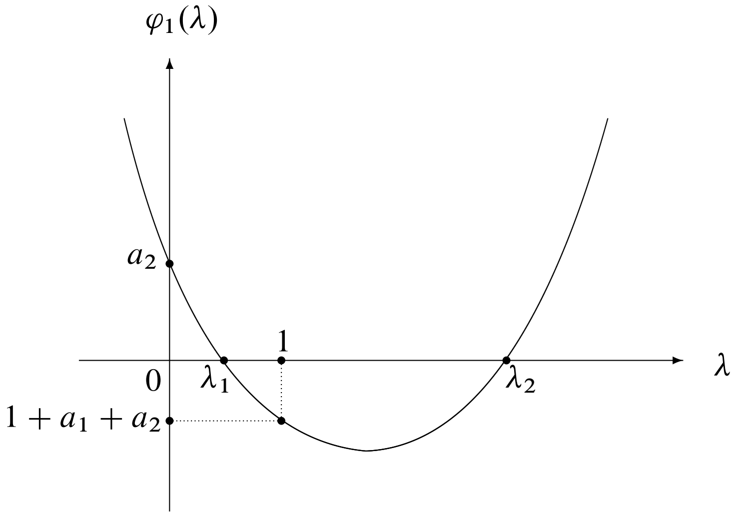

FIGURE 5. Case of $0 < \gamma_1 < 1$.

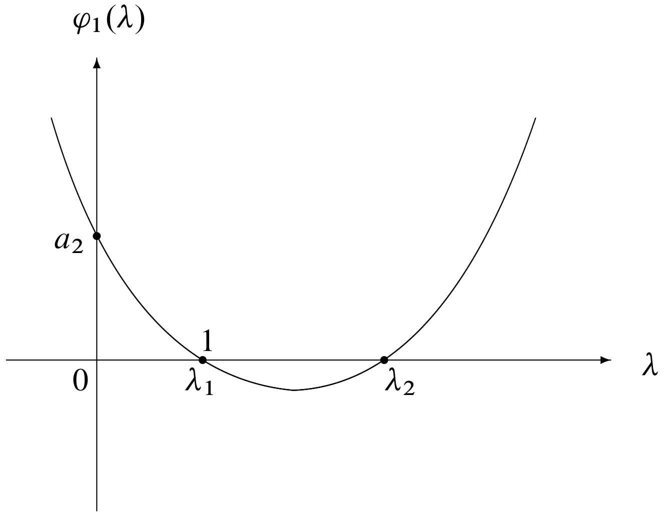

FIGURE 6. Case of $\gamma_1 = 1$.

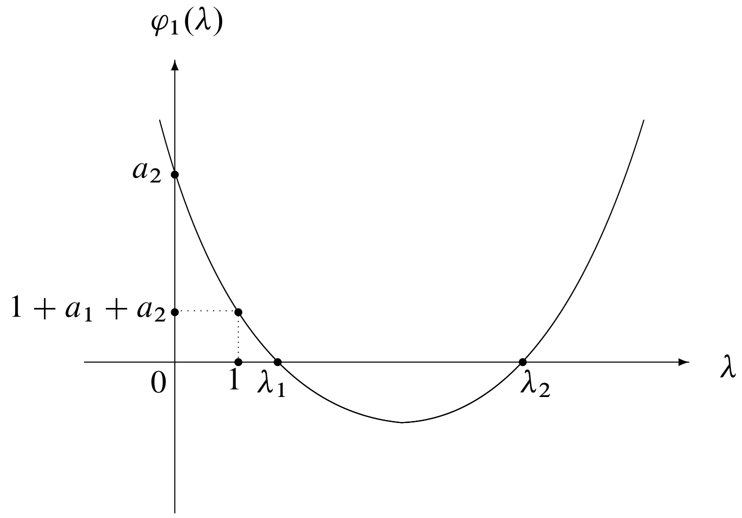

FIGURE 7. Case of $1 < \gamma_1 < 1 + \zeta$

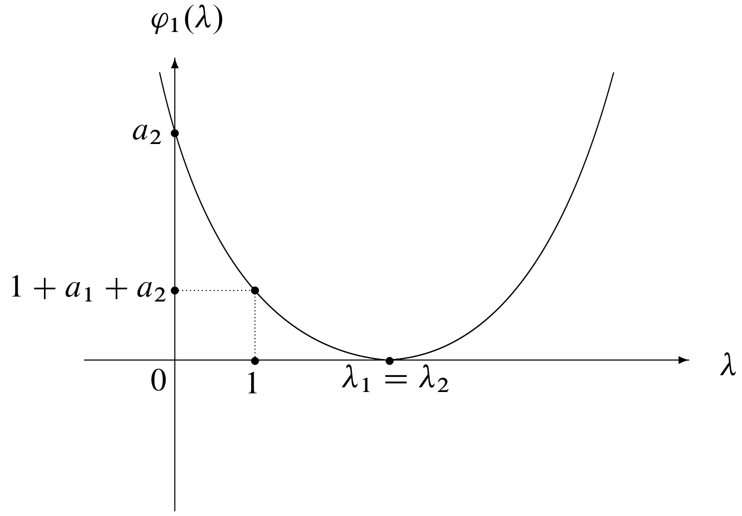

FIGURE 8. Case of $\gamma_1 = 1 + \zeta$

Appendix C. Hopf Bifurcation Theorem

Proposition 6.3 is a direct consequence of the following version of the Hopf bifurcation theorem (see Asada, Chiarella, Flaschel and Franke 2003, 2010, Mathematical appendices, and Gandolfo 2009, Ch.24).

THEOREM C.1. *Let $\dot{x} = f(x;\varepsilon)$, $x \in R^n$, $\varepsilon \in R$ be a system of differential equations with a parameter ε. Suppose that the following properties* (i)–(iii) *are satisfied.*

(i) *This system has a smooth curve of equilibria given by $f(x^*(\varepsilon);\varepsilon) = 0$.*

(ii) *The characteristic equation $|\lambda I - Df(x^*(\varepsilon_o);\varepsilon_o)| = 0$ has a pair of pure imaginary roots, $\lambda(\varepsilon_o)$ and $\overline{\lambda}(\varepsilon_o)$, and no other roots with zero real parts, where $Df(x^*(\varepsilon_o);\varepsilon_o)$ represents the Jacobian matrix of the above system at $(x^*(\varepsilon_o), \varepsilon_o)$.*

(iii) $\left.\dfrac{d\,\mathrm{Re}\,\lambda(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\varepsilon_o} \neq 0$, *where* Re $\lambda(\varepsilon)$ *is the real part of* $\lambda(\varepsilon)$.

Then, there exists a continuous function $\varepsilon(\gamma)$ with $\varepsilon(0) = \varepsilon_o$, and for all sufficiently small values of $\gamma \neq 0$ there exists a continuous family of non-constant periodic solutions $x(t,\gamma)$ for the above dynamic system, which collapses to the equilibrium point $x^*(\varepsilon_o)$ as $\gamma \to 0$. The period of the cycle is close to $\dfrac{2\pi}{\mathrm{Im}\,\lambda(\varepsilon_o)}$, where Im $\lambda(\varepsilon_o)$ *is the imaginary part of* $\lambda(\varepsilon_o)$.

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