Post Keynesian Review
Vol. 1, No. 1/2, Pages 1–17 (the 10th December, 2012)
ISSN 2187-4875

Asymptotic Stability of Pasinetti and Dual Equilibria in a Growth Model with Robinson Type Investment Function and Keynes-Wicksell Adjustment Process

TORII, Akio

ABSTRACT. Samuelson-Modigliani developed the duality theorem by applying the neoclassical growth model. We should now ask whether the duality theorem is intrinsic to the macro growth theory or it depends on the neoclassical growth model. This paper examines the dual theorem rather in the Keynesian world. Our model assumes an independent Robinsonian investment function and the Keynes-Wicksel adjustment process. Our main result is: if the adjustment speed of the product price is faster compared to that of money wage rate, then the duality theorem does not hold for any magnitude of saving propensity and the shape of production function, while a dynamically stable steady growth path with full employment is, if ever exists, only the path that supports Pasinetti theorem.

Keywords: Pasinetti's theorem, Robinsonian animal spirits, Keynes-Wicksell adjustment, asymptotically stable

1. Pasinetti's Theorem and the Duality Theorem

1.1. Pasinetti's theorem vs. the duality theorem. Pasinetti(1962) presented a proposition that asserts: *On the full employment steady state growth path, the rate of profit does not depend on the rate of saving of the working class.* This proposition was derived by the Kaldorian effective demand principle and the necessary conditions for the steady state growth path. The model employed for the proof was, however, not complete to determine production and investment at each time point. The effective demand principle works only in the equality of investment and saving, and the independence of the investment from the saving does not work in full in his proof. Thus, the effective demand principle is not crucial for Pasinetti's theorem.

Professor Dr. Torii Akio, Faculty of Economics, Chuo University, Tokio, Japan.
The prototype of this article was published as Torii(1987).

Indeed, Pasinetti's theorem can be derived simply from the necessary condition for the steady state growth path. The amount of capital owned by the capitalist class, K_c, brings about the profit income, rK_c, where r is the rate of profit. Since the capitalist class saves s_c of income, capital increases by $s_c r K_c$. Hence, their capital is accumulated by the rate of $s_c r$. In the steady state growth path, all real variables grow in the same rate, so that $s_c r$ is equal to the rate of labour growth n. Then, one obtains $r = \frac{n}{s_c}$. This is Pasinetti's theorem. Since Pasinetti's theorem is derived from the Kaldorian distribution basis, in what follows we call it the Kaldor-Pasinetti model, in short K-P, or K-P model.

Pasinetti's theorem left a point to be discussed, thus. Mead (1966) and Samuelson-Modigliani(1966), abbreviated as S-M in the following, argued as follows.

Let K denote the total amount of capital, and the dot differentiation by the time. Differentiating $\frac{K_c}{K}$ with respect to time, one obtains $\frac{K_c}{K}\left(\frac{\dot{K}_C}{K_C} - \frac{\dot{K}}{K}\right)$. Since proportions of real variables never change on the steady state growth path, the differentiation should be 0. $\frac{\dot{K}}{K}$ is the rate of accumulation of total capital, and it should be equal to n for full employment. Hence, it follows that the necessary condition for the steady state growth path is reduced to

$$(s_c r - n)\frac{K_c}{K} = 0. \tag{1.1}$$

The set of solutions of this equation includes $\frac{K_c}{K} = 0$, in addition to $s_c r = n$. Thus, there are 2 types of the steady state growth path: a path that justifies the Pasinetti Theorem and a path with $\frac{K_c}{K} = 0$. The former path is denoted as P-path and the latter as D-path throughout this paper. On D-path the relation $s_c r = n$ need not be hold. S-M claimed that both P-path and D-path can be the steady state path depending on the values of parameters. They called their conclusion the duality theorem. The duality theorem looks more comprehensive. Is it really so?

1.2. Framework of the analyses. In order to prove the duality theorem S-M employed a neoclassical growth model. The neoclassical growth model relies on the marginal productivity theory in determining the rate of profit. The mechanism is completely different from Kaldor's theory of distribution (Kaldor, 1956) that underlies Pasinetti's theorem. What we consider is whether the marginal productivity theory which replaces Kaldor's distribution theory makes the duality theorem valid, or Pasinetti's original model is not free from the possibility of the stable D-path. In other words, the problem is to what an extent the duality theorem depends on the framework of the neoclassical growth model.

Needless to say, the necessary condition (1.1) cannot avoid the solution $\frac{K_C}{K} = 0$. The problem is whether D-path is asymptotically stable or not. S-M showed that D-path is asymptotically stable under some set of parameters and a class of production functions. Hence, our task is to check the stability of D-path in Pasinetti's original framework without changing any basic settings.

One of the common features of Keynesian models is the independent investment function: the effective demand principle asserts that the investment is determined independent of the determination of the savings. Thus, how much the duality theorem is intrinsic to the neoclassical growth model rests on the asymptotic stability of D-path in a growth model equipped with an independent investment function. To check the asymptotic stability is the main task of this paper.

1.3. Stability of the growth path and the adjustment process. In the Kaldorian theory of distribution, distribution changes so as to raise savings which is equivalent to the investment. This variable rate of savings characterizes Kaldor's model, that dealt with the instability in Harrod's model. In order for savings to change in accordance with changes in distribution, there should exist multiple classes of agents who have different propensities to save: the model needs the capitalists.

If there is no class of agents with higher saving rate, *i.e.* $\frac{K_c}{K} = 0$, then Kaldor's distribution mechanism can never work, so that the rate of savings will be fixed. Hence, to keep the stability of D-path needs a distribution mechanism which replaces the Kaldor's mechanism of distribution and adjusts the savings rate. Without such a mechanism the duality theorem does not hold in Kaldor's world. Conversely, if an appropriate mechanism of distribution is set up and a short-run equilibrium is secured to exist, then there is a possibility that the D-path is asymptotically stable for some set of parameters and a class of the production function.

The S-M model is a neoclassical model with variable capital coefficient and Say's law which ensure the long-run stability. Irrespective of the existence of capitalists, in other words, even if the rate of savings is not variable, the economy is flexible enough to secure the stability of D-path.

Darity(1981) has introduced the dynamic distribution model with an independent investment function. The investment function had three variables: the marginal productivity of capital, the rate of interest, and the level of employment. Although he analyzed the stability of the steady state growth path, his model is not appropriate to examine the duality because he introduced the monetary authorities that function as an adjusting mechanism. The stability is ensured by the assumption even if no capitalists exist. His model will be examined later in this paper.

As is shown in the following analysis, the stability of the growth path, the conclusion, depends on the setting of the short-run economic framework. This is inevitable because what matters here is the stability of D-path in the economy which embodies the Kaldorian distribution theory.[1]

The plan of the paper is as follows: Section 2 will be dedicated to the setting of our model. Section 3 will focus on the equilibrium points of the model and their asymptotic stability, and presents three propositions with their proof.

2. The Model

2.1. Variables, parameters and basic relations. The variables and constants that are employed in the following will be defined as follows:

Let Y, K_c, K_w, K, I and L denote, respectively, the real level of production, the amount of capital owned by the capitalist class, the amount of capital owned by the working class, the total amount of capital, the level of investment, and the amount of labour. Let P be the nominal price of products, W be the money wage rate, and r be the rate of profit. All those are variables.

As for constants: n, s_w, s_c, α and β denote respectively the growth rate of labour, the saving rate of the working class, the saving rate of the capitalist class, the adjustment speed of prices in the products market, and the adjustment speed of money wages in the labour market.

Variables are transformed into simpler forms as follows:

$$k \equiv \frac{K}{L}, \;\; t \equiv \frac{K_c}{K}, \;\; y \equiv \frac{Y}{K}, \;\; w \equiv \frac{W}{P}. \tag{2.1}$$

In addition, one has:

$$K = K_c + K_w. \tag{2.2}$$

[1]Morishima(1977) mentioned the importance of price mechanism for the validity of Pasinetti theorem.

It is assumed that the economy is equipped with a well-behaved production function $Y = F(K, L)$, that is homogeneous with degree 1:

$$Y = F(K, L) = L \cdot F\left(\frac{K}{L}, 1\right) \equiv Lf(k). \tag{2.3}$$

This defines the function $f(\cdot)$. One obtains that

$$f(k) = w + rk. \tag{2.4}$$

We assume the production function well behaved:

$$y(k) \equiv \frac{Y}{K} = \frac{f(k)}{k} > f_k, \ \lim_{k \to 0} \frac{f(k)}{k} = \infty, f(0) > 0. \tag{2.5}$$

K-P took up the steady state growth path with full employment. The assumption of full employment affects much to the setting of the model, so that some preliminary considerations are necessary before considering the price adjustment process that is related to investment and saving. Full employment considered in K-P is the growth path in which the rate of investment exceeds the rate of labour growth, so that labour demand is always greater than labour supply.[2] It is the "restricted golden age"(Robinson(1962)). Thus, we confine the following analysis to full employment induced by the bottleneck caused by low natural growth rate. K-P assumed that in such a situation price adjustment is faster than changes in demand, and analyzed the adjustment mechanism of distribution. If the assumption of full employment is removed, price adjustment will not be relatively fast enough, so that one should introduce the quantity adjustment mechanism with unemployment into the model. Then, distribution is adjusted by a different mechanism. Darity's model assumes that the monetary authority takes appropriate measures so as to realize full employment.

In the neighbourhood of the steady state growth path with full employment, the condition that labour demand exceeds labour supply is equivalent to the condition that planned investment exceeds savings, as will be shown in the next section. Therefore, if full employment in the above meaning is assumed, then disequilibrium that *ex ante* investment is always greater than *ex post* savings arises. In this paper, in order to cope with this disequilibrium, it is assumed that part of investment exceeding savings will never be realized. In other words, only the part that matches *ex ante* savings is realized. This assumption is highly neoclassical, but innocent: it does not affect the conclusion. This is because, irrespective of adjustment of short-run disequilibrium, the price adjustment mechanism determines the property of the model. Even

[2] See Pasinetti(1974), pp.100-1.

if one assumes, for example, that a certain part of investment and/or savings is realized, the qualitative nature of the conclusion will not be altered. Remark that under this assumption, consumption of the working class is realized, so that it is consistent with full employment.

In the next place, take savings and capital accumulation. Labourers save s_w of wage income wL and profit income rK_w, whilst capitalists do s_c of profit income rK_c. From the assumption mentioned above, savings are all realized, so that capital accumulation is expressed by:

$$\dot{K}_w = s_w(wL + rK_w), \tag{2.6}$$

$$\dot{K}_c = s_c r K_c. \tag{2.7}$$

The total of savings in the economy is given by $s_w(wL + rK_w) + s_c rK_c$.

If *ex ante* investment exceeds savings, the product price rises in proportion to the difference between investment and savings. Let α denote the adjustment speed, and

$$\frac{\dot{P}}{P} = \alpha \frac{I - \{s_w(wL + rK_w) + s_c rK_c\}}{K}. \tag{2.8}$$

Under this price adjustment mechanism, changes in prices should be faster than changes in demand, especially investment, so that the rate of profit varies. This price adjustment mechanism plays a central role in the K-P distribution mechanism, and is called the Keynes-Wicksell adjustment by Stein(1966, 1970) that discussed the adjustment of disequilibrium between savings and investment in a monetary growth model.

In the third place, consider the adjustment of money wages in the labour market. Labour supply increases with a fixed rate (natural rate of growth n), that is

$$\dot{L} = nL. \tag{2.9}$$

Full employment is defined as a state with labour demand exceeding labour supply. Let X_d denote for labour demand. Since the well-behaved production function is assumed, one has $w = F_L(K, X_d)$. Let $w^* = F_L(K, L)$, and, in the neighbourhood of full employment, one obtains

$$X_d - L \approx w^* - w \approx F_L(K, L) - w.$$

Therefore, the full employment assumption is represented by:

$$F_L(K, L) \geq w. \tag{2.10}$$

Remark that the condition (2.10) does not restrict any basic parameters of the model. It restricts only the equilibrium to confine the analysis within full employment situation.

The labour market is adjusted in such a way that the money wage rate increases in proportion to the difference between labour demand and supply $X_d - L$; let β denote the adjustment speed, and one obtains

$$\frac{\dot{W}}{W} = \beta(F_L - w). \tag{2.11}$$

With a well-behaved production function, the marginal productivity is introduced in (2.10) and (2.11). Remark, however, that except the case $\beta \to \infty$ the setting alone does not determine the rate of profit.

Ex post investment is restricted by savings, (2.6) and (2.7), and the above (2.10) and (2.11) are all deeply coloured by neoclassical tone, but these are employed here just to make model analysis clear. Even with these assumptions and settings, the rate of profit and distribution change in accordance with wage rise and increases in product prices. Therefore, the rate of profit is simultaneously determined with the equilibrium in the product market, where the effective demand principle governs. Distribution is not determined by marginal productivity alone.

As for the investment function, we employ the Robinson type animal spirits investment function. The higher the rate of profit, the greater the capitalists' motivation for investment. In the "restricted golden age" of Robinson(1962), it is suggested that investment is controlled by the rate of interest when the investment motivation is not fully satisfied. Since our argument here is confined to the real terms, it is assumed, as suggested by Robinson, that investment is held down if employment is restricted, that is, if $F_L > w$ or $r > f_k$. Since f_k is a decreasing function of k, f_k deviates more from r and suppresses the investment, as k increases. It is also assumed that the effect of animal spirits is diminishing. Furthermore, it is assumed that they require a positive profit rate, r_o, ensured to implement the investment. This minimum rate may be interpreted as the smallest risk premium.[3] Thus, one obtains the following investment function:

$$I \equiv g(r, k)K, \tag{2.12}$$

[3] K-P also assumed the existence of the least profit rate for motivating investment. See Pasinetti(1974), p.101, note 26.

with

$$(2.13) \qquad g_r > 0,\ g_{rr} < 0,\ g_k < 0,\ \text{if}\ \ r > r_o;\ \ g = 0,\ \text{if}\ \ r \leq r_o,$$

where $r_o > 0$. This investment function is the simplest with the characteristics common to Keynesian models. The capitalists behind this investment function are active entrepreneurs: it is faithful to K-P.

2.2. Comments on Darity. The above model differs from Darity's most remarkably in the adjustment mechanism. Since the working of the adjustment mechanism affects the choice of investment functions, some comments will be made on Darity's model. Darity introduced the investment function of the form

$$I = g(f_k, r, e)K,$$

with $g_{f_k} < 0$, $g_r < 0$ and $g_e > 0$, where r is a rental price of capital, so that it can be interpreted as the rate of interest, and e denotes the level of employment.

Monetary authorities adjust the rate of interest to achieve full employment:

$$\dot{r} = \gamma(e - 1).$$

The level of employment is determined by the equilibrium in the product market

$$I(f_k, r, e) = S(k, e, t, r),$$

where S stands for savings of the economy as a whole. It is assumed that $I_e < S_e$. From this assumption, in the neighbourhood of full employment $e = 1$,

$$e \gtrless 1 \Longleftrightarrow I(f_k, r, e = 1) \gtrless S(k, e = 1, t, r),$$

where k, r and t are historically given. Therefore, the adjustment process is rewritten as

$$\dot{r} \approx I(f_k, r, e = 1) - S(k, e = 1, t, r).$$

As pointed out previously, whether the dual theorem holds or not depends on the existence of stable equilibrium in the economy without capitalists, that is, in the economy in which the rate of savings of the economy does not change in accordance with changes in distribution. In Darity's model, it is assumed that if there is no capitalist class ($t = 0$), savings do not depend on r that reflects distribution:

$$\dot{r} \approx I(f_k, r, e = 1) - S(k, e = 1, t = 0).$$

This is his adjustment rule on r. The process makes investment equal to savings and brings about a stable equilibrium, because $I_r < 0$.[4] Hence, in the neighbourhood of $t = 0$, Darity's model shows similar characteristics with the S-M neoclassical growth model. Therefore, it is inappropriate to analyze the relationship between the K-P theorem and neoclassical growth model. In our model, the stability of equilibria depends on the setting of the investment function, and not on something else.

2.3. Summary of the model. In this paper, we employ a simple investment function, as described above. Our investment function will not automatically ensure the stability of equilibrium. r remains to be the rate of profit.

The framework of our analysis is now complete. Variables that are controlled by (2.3)–(2.13) are $K,\ K_w,\ K_c,\ P,\ W$ and L, on the given initial values of $Y,\ r,\ g(r)$ and F_L.

By transforming variables in view of (2.1)–(2.2), one obtains

$$\begin{aligned} \dot{r} &= (f_k - r)\{s_w y(k) + rt(s_c - s_w) - n\} - (y(k) - r)\times \\ &\quad [\beta k(r - f_k) - \alpha\{g(r,k) - s_w y(k) - rt(s_c - s_w)\}]\,, \end{aligned} \tag{2.14}$$

$$\dot{t} = t\{s_c r - s_w y(k) - rt(s_c - s_w)\}, \tag{2.15}$$

$$\dot{k} = k\{s_w y(k) + rt(s_c - s_w) - n\}. \tag{2.16}$$

3. Stability of Equilibria

This section analyses the stability of equilibria determined by the system of equations (2.14)-(2.16). Equilibria are solutions of $\dot{r} = \dot{t} = \dot{k} = 0$:

$$\begin{aligned} &(f_k - r)\{s_w y(k) + rt(s_c - s_w) - n\} - (y(k) - r)\times \\ &\quad [\beta k(r - f_k) - \alpha\{g(r,k) - s_w y(k) - rt(s_c - s_w)\}] = 0, \end{aligned} \tag{3.1}$$

$$t\{s_c r - s_w y(k) - rt(s_c - s_w)\} = 0, \tag{3.2}$$

$$k\{s_w y(k) + rt(s_c - s_w) - n\} = 0. \tag{3.3}$$

From (3.3), one gets $k = 0$ or

$$s_w y(k) + rt(s_c - s_w) - n = 0. \tag{3.4}$$

[4]More rigorously, one has to check the stability of solutions of the system of simultaneous equations, *i.e.* the monetary adjustment and capital accumulation. Capital accumulation is expressed by

$$\dot{k} = k\{g(k, r, e) - n\}.$$

It is easily seen that the stability condition for $t = 0$ is $g_r < 0$.

Remark that $k = 0$ is not an asymptotically stable solution, because, in view of assumption (2.5), $y(k)$ is sufficiently high in the neighbourhood of $k = 0$, which implies $\dot{k} > 0$ from (2.16). Hence, $k = 0$ cannot be an asymptotically stable solution. Therefore, (3.4) gives the asymptotically stable solution that satisfies (3.3).

Rewrite (3.1) and (3.2) by using (3.4), and one obtains:

$$(y - r)[\beta k(r - f_k) - \alpha\{g(r) - n\}] = 0, \tag{3.5}$$

$$t(s_c r - n) = 0. \tag{3.6}$$

Meanwhile, from (2.4), and $w > 0$, one has

$$y > r. \tag{3.7}$$

Hence, (3.5) implies

$$\beta k(r - f_k) = \alpha\{g(r, k) - n\}. \tag{3.8}$$

The condition of full employment (2.10) is rewritten, in view of (2.4) and (3.8), as

$$r \geq f_k. \tag{3.9}$$

Therefore, in the neighbourhood of the steady state growth path,

$$g(r, k) \geq n, \tag{3.10}$$

from (3.8) and (3.9).

The point that $g \geq n$ is equivalent to $w \leq F_L$, mentioned in Section 2, is clear from the above mathematical manipulation.

Now, in view of (3.6), one sees that there are 2 equilibria: the one with $t = 0$ and the other with $s_c r = n$. $t = 0$, (3.4), and (3.8), give an equilibrium, D-path, and $s_c r = n$, (3.4), and (3.8), give another equilibrium, P-path. On D-path, r depends on s_w, whilst on P-path, r depends solely on s_c and n.

If both of two paths can be asymptotically stable depending on the values of the parameters, the duality theorem is valid. If only P-path is asymptotically stable, then Pasinetti's theorem is true. Then, as a first step, we assume the existence of equilibria and check the conditions for asymptotic stability.

By linearizing (2.14)–(2.16) in the neighbourhood of equilibrium, one gets the following Jacobi matrix:

$$J = \begin{array}{c} \begin{array}{ccc} \qquad A \qquad & \qquad\qquad B \qquad\qquad & \qquad C \end{array} \\ \begin{pmatrix} t\{s_c - t(s_c - s_w)\} & s_c r - s_w y(k) - 2rt(s_c - s_w) & \frac{-s_w t(f_k - y)}{k} \\ kt(s_c - s_w) & kr(s_c - s_w) & s_w(f_k - y) \end{pmatrix} \end{array}$$

where,

$$\begin{aligned}
A &= (f_k - r)t(s_c - s_w) - (y - r)\{(\beta k - \alpha g_r) + \alpha t(s_c - s_w)\},\\
B &= r(s_c - s_w)\zeta,\\
C &= s_w(f_k - y)\zeta - (y - r)\{\beta k(r - f_k - kf_{kk}) - \alpha g_k\},
\end{aligned}$$

with $\zeta = f_k - r - \alpha(y - r)$.

First we check the stability of D-path assuming its existence. When $t = 0$, then the Jacobi matrix is

$$J_D = \begin{pmatrix} (r - y)(\beta k - \alpha g_r) & B & C \\ 0 & s_c - rs_w y(k) & 0 \\ 0 & k_r(s_c - s_w) & s_w(f_k - y) \end{pmatrix}.$$

The characteristic equation $|\lambda E - J_D| = 0$ is reduced to

$$\{\lambda - (r - y)(\beta k - \alpha g_r)\}\{\lambda - (s_c r - s_w y(k))\}\{\lambda - s_w(f_k - y)\} = 0.$$

One obtains 3 real roots:

$$\lambda = (r - y)(\beta k - \alpha g_r),\ s_c r - s_w y(k),\ s_w(f_k - y).$$

The local stability of the dynamic system along D-path requires all roots be negative. With (2.5) and (3.7), one obtains equivalent conditions:

$$s_c r - s_w y(k) < 0, \tag{3.11}$$

$$\beta k - \alpha g_r > 0. \tag{3.12}$$

Since $t = 0$, from (3.4), one has

$$s_w y(k) = n. \tag{3.13}$$

Hence, (3.11) is equivalent to

$$s_c r < n. \tag{3.14}$$

Next, we check the stability of P-path. In view of (3.4), the Jacobi matrix becomes:

$$\begin{pmatrix} A & B & C \\ t\{s_c - t(s_c - s_w)\} & -rt(s_c - s_w) & -\frac{s_w t(f_k - y)}{k} \\ kt(s_c - s_w) & kr(s_c - s_w) & s_w(f_k - y) \end{pmatrix},$$

and its characteristic equation is reduced to

$$\lambda^3 + \gamma_2\lambda^2 + \gamma_1\lambda + \gamma_o = 0,$$

where

$$\gamma_2 = \delta_1 + \{\beta k - \alpha g_r + \alpha t(s_c - s_w)\}\delta_2, \tag{3.15}$$

$$\gamma_1 = \delta_3 + \alpha\delta_4 + \beta\delta_5 + (\beta k - \alpha g_r)\delta_6, \tag{3.16}$$

$$\gamma_0 = \beta r s_c \delta_5, \tag{3.17}$$

and

$$\begin{aligned}
\delta_1 &= n - s_w f_k + (r - f_k)t(s_c - s_w) \\
&> n - s_w f_k > n - s_w y > rt(s_c - s_w), \\
\delta_2 &= y - r, \\
\delta_3 &= r s_c t(s_c - s_w)(r - f_k), \\
\delta_4 &= r s_c t(s_c - s_w)(y - r), \\
\delta_5 &= kt(s_c - s_w)(r - f_k - k f_{kk})(y - r), \\
\delta_6 &= (y - r)(n - s_w f_k).
\end{aligned}$$

One can check easily that $\delta_i > 0$ $(i = 1, \cdots, 6)$ with (2.3), (3.4), (3.7), and (3.9).

The Routh-Hurwitz theorem tells that necessary conditions for the local stability of the system along P-path are,

$$\gamma_2 > 0,\ \gamma_1 > 0,\ \gamma_o > 0,\ \gamma_2\gamma_1 > \gamma_o.$$

From the stability conditions of the dynamic system along the two equilibrium paths, one obtains the following 3 propositions that clarify the relations between Pasinetti's theorem and the duality theorem.

PROPOSITION 3.1. *Suppose* $\dfrac{\beta}{\alpha} \to 0$. *If D-path is asymptotically stable, it is necessary that $r \to \infty$ and $g_r \to 0$ in equilibrium. This conclusion is true irrespective of values of s_c, s_w and n.*

Proof. From (3.10), (3.11) and (3.13), for the stability of D-path,

$$g(r,k) \geq n = s_w y(k) > s_c r \tag{3.18}$$

should hold. From (2.13), however, if $0 < r \leq r_o$, then $g(r,k) = 0 < s_c r$, so that (3.18) does not hold. Thus, the condition (3.18) requires $r > r_o$, irrespective of the value of s_c. With (3.7), it follows that

$$y(k) > r > r_o. \tag{3.19}$$

Since $y(k)$ is a decreasing function, the set of k satisfying (3.19) is bounded from the above. Let $\bar{k} \leq y^{-1}(r_o)$ denote the supremum, and, in view of the boundedness of k, one has

$$\frac{\beta}{\alpha} > \frac{g_r(r,k)}{k} > \frac{g_r(r,\bar{k})}{\bar{k}} \tag{3.20}$$

from (3.12).

When $\frac{\beta}{\alpha} \rightarrow 0$, $g_r \rightarrow 0$ is necessary. Since we have assumed $g_r > 0$ for finite value of r as (2.13), $r \rightarrow \infty$ is required for $g_r \rightarrow 0$. □

The equilibrium depends on parameters like rates of savings. The proposition above restricts the profit rate at the equilibrium. In order for the rate of profit to be infinite, $k \rightarrow 0$ should hold from (3.7). Then, by (3.18) $s_w \rightarrow 0$ and $s_c \rightarrow 0$ should follow. The converse, however, is not true: $s_c \rightarrow 0$ is not sufficient for D-path to be stable when $\beta \rightarrow 0$. Note that s_w, s_c and n do not appear in the condition (3.20).

Consider a possible assumption on the investment function,

$$g_r(r,k) > \varepsilon > 0,$$

for all $r > r_o$ where ε is a positive constant. Under this assumption, entrepreneurs maintain their investment motivation by animal spirits when the positive minimum rate of profit is guaranteed. In other words, investment is never saturated within the scope of analysis; since capitalists save a fixed part of the profit income, it is hard to imagine that increases in profit will not result in a greater amount of investment. This assumption keeps the Keynesian nature of the model. Thus, if we introduce a lower bound for g_r, the right-hand side of (3.20) has a positive lower bound. Therefore, for the duality theorem to hold, the adjustment speed of money wages should surpass a certain level in comparison with the price adjustment speed of the product market. If this value of the lower bound depends on the parameters s_c, s_w and n, then one can confirm the validity of the duality theorem that asserts the convergence of equilibrium either to $t = 0$ or to $s_c r = n$, depending on the parameters. The proposition asserts that it is not true. If the price adjustment speed in the product market is relatively high, then for any rates of savings, and for any rate of labour growth, D-path with $t = 0$ can never be asymptotically stable.

Apart from such a new assumption, Proposition 3.1 shows that in order for the profit rate to be finite in equilibrium, the lower bound, that does not depend on parameters, exists for the ratio $\frac{\beta}{\alpha}$. This proposition depends only on the assumption on risk premium and animal spirits. This contrasts much to Darity's model in which $g_r < 0$ plays an crucial role.

Contrast to the duality theorem, Pasinetti's theorem holds even when the adjustment speed of money wages is relatively slow. The next proposition states this.

PROPOSITION 3.2. *For an arbitrary positive number α, there exists a pair of (s_c, s_w, n) such that P-path exists and asymptotically stable, even for the case $\beta \to 0$.*

Proof. (1) The first half of the proof is dedicated to the proof of the existence of equilibrium path satisfying $s_c r = n$ when $(\alpha, \beta) = (\alpha^*, 0)$, by choosing an appropriate pair (s_c, s_w, n).

For an arbitrarily chosen α^*, consider the case $(\alpha, \beta) = (\alpha^*, 0)$. Put $\beta = 0$ in (3.1)–(3.3), and the condition for $t \neq 0$ becomes

$$g(r, k) = s_w y(k) + rt(s_c - s_w) = s_c r = n. \tag{3.21}$$

Firstly, consider a special case of $s_w = 0$. Then, (3.21) becomes

$$g(r, k) = s_c r = n, \; t = 1. \tag{3.22}$$

Take a positive k^*, and consider an equation $g(r, k^*) = s_c r$ with respect to r. From (2.13), it follows that $g(r_o, k^*) = 0 < s_c r$. This and $g_{rr} < 0$ enable us to take a value of s_c in such a way that $g(r, k^*) = s_c r$ has at least 2 positive solutions. Let $s_c{}^*$ be one such value. Let r_1 and $(r_1 <)r_2$ be the minimum and the maximum among solutions, respectively, and it follows

$$g_r(r_1, k^*) > s_c{}^*, \; g_r(r_2, k^*) < s_c{}^*.$$

Let r_2 be r^*. One may choose k^* so as to ensure $y(k^*) > r^* > f_k(k^*)$.[5] Hence, it is obvious that, at the point $(s_c, s_w, n) = (s_c{}^*, 0, s_c{}^* r^*)$, $(r^*, k^*, 1)$ is a solution of (3.22).

Next, consider a solution at $(s_c{}^*, s_w{}^{**}, s_c{}^* r^*)$, which locates in the neighbourhood of $(s_c, s_w, n) = (s_c{}^*, 0, s_c{}^* r^*)$. One can check that $(r, k, t) = (r^*, k^*, t^{**})$, where $t^{**} = \frac{s_c{}^* r^* - s_w{}^{**} y(k^*)}{s_c{}^* r^* - s_w{}^{**} r^*}$ (< 1 because of $y > r$), is a solution of (3.21). In fact, by choosing sufficiently small $s_w{}^{**}$, one can get t^{**} that is sufficiently close to 1; hence, in view of $g_r(r^*, k^*) < s_c{}^*$, it follows that

$$g_r(r^*, k^*) < t^{**}(s_c{}^* - s_w{}^*). \tag{3.23}$$

[5] In fact, the existence of such a k^* is guaranteed as in the following: Take k^* sufficiently small, and the region $(y(k^*), f_k(k^*))$ shifts upward. For such a k^*, one can take sufficiently small $s_c{}^*$, and hence sufficiently large r^*, so that r^* lies inside of $(y(k^*), f_k(k^*))$.

Thus one can choose (s_c, s_w, n) which ensures the existence of equilibrium path satisfying $s_c r = n$ when $(\alpha, \beta) = (\alpha^*, 0)$.

(2) Now, we complete the proof. When $\beta > 0$, the proof above for the case $\beta = 0$ is still valid and the existence of the equilibrium is ascertained for sufficiently small value of β.

For a positive value of β, (3.23) is still true, if the value is sufficiently small. Then, the coefficients of β in (3.15) $\approx$(3.17) are all positive. Thus, positiveness of coefficients of the characteristic equation is fulfilled for sufficiently small β.

From (3.15)

$$\gamma_2 = n - s_w f_k + (r - f_k)t(s_c - s_w) + (\beta k - \alpha(g_r - t(s_c - s_w)))\delta_2,$$

then one may choose s_w in such a way that $\gamma_2 > n$, as a solution for sufficiently small s_w is considered here. Furthermore, one gets $\gamma_1 > \beta\delta_5,\ \gamma_o = \beta r s_c \delta_5 = n\beta\delta_5$. Hence, $\gamma_2\gamma_1 > n\beta\delta_5 = \gamma_o$. Thus, stability is ensured for a sufficiently small positive value of β. □

Proposition 3.2 proves the validity of Pasinetti's theorem in an extreme case with $\frac{\beta}{\alpha} \to 0$ in order to contrast Pasinetti's theorem with the duality theorem. In view of the proof, it is clear that one can choose s_c in such a way that r satisfying $g(r, k) = s_c r$ is finite. Recall that the D-path is not asymptotically stable for any parameters when r is finite in the case $\beta \to 0$.

The last proposition demonstrates that Pasinetti's theorem is valid for the wider range of parameters than the duality theorem.

PROPOSITION 3.3. *Suppose that there exists an asymptotically stable D-path for arbitrary chosen (α, β) and $({s_c}^D, {s_w}^D, n^D)$. Then, one can choose (s_c, s_w, n) so that there exists an asymptotically stable P-path for the same (α, β).*

Proof. Let (r^D, k^D, t^D) be the solution, one has

$$ {s_c}^D r^D < n^D, \tag{3.24}$$

$$ \beta k^D - \alpha g_r(r^D, k^D) > 0. \tag{3.25}$$

from the stability condition of D-path, (3.12) and (3.14). Let $r^P = \frac{r^D}{{s_c}^D}$ and it is easy to see $r^P > r^D$.

Next, take (r, k) that fulfills (3.8). By differentiating (3.8), one obtains

$$\frac{dk}{dr} = -\frac{\beta k - \alpha g_r}{\beta(r - f_k - k f_{kk}) - \alpha g_k}, \tag{3.26}$$

where the denominator is positive in view of $r - f_k > 0,\ f_{kk} < 0$ and $g_k < 0$. At (r^D, k^D), $\dfrac{dk}{dr} < 0$ from (3.25). Let k^P denote the value of k obtained from (3.8) with $r = r^P$, and it follows that

$$\beta k^P - \alpha g_r(r^P, k^P) > 0. \tag{3.27}$$

In fact, if (3.27) does not hold, at an r^* such that $r^D < r^* < r^P$ $\beta k = \alpha g_r$ on the curve (3.8). Hence, $\left.\dfrac{dk}{dr}\right|_{r=r^*} = 0$. In addition, $\dfrac{dk}{dr} > 0$ at $r = r^P$. Therefore, k fulfilling (3.8) needs to be a local minimum at $r = r^*$. In the light of

$$\frac{d^2k}{dr^2} = \frac{\alpha g_{rr}}{\beta(r - f_k - k f_{kk}) - \alpha g_k},$$

however, the denominator of this at $r = r^*$ is negative by assumption. This contradicts that k is a local minimum at $r = r^*$. Therefore, (3.27) is true. One sees also from the above that for $r^D < r < r^P$, $\dfrac{dk}{dr} < 0$. Hence, $k^P < k^D$.

One may take a sufficiently small $s_w{}^P$ in such a way that $n^D > s_w{}^P y(k^P)$. If t^P is set as $t^P = \dfrac{s_c{}^D r^P - s_w{}^P y(k^P)}{s_c{}^D r^P - s_w{}^P r^P}$, then $t^P < 1$. This is because $s_c{}^D r^P - s_w{}^P y(k^P) > n^D - s_w{}^P y(k^P) > 0$, $s_c{}^D r^P - s_w{}^P y(k^P) = (s_c{}^D - s_w{}^P) r^P > 0$, and $y(k^P) > y(k^D) = \dfrac{n^D}{s_w{}^D} > \dfrac{n^D}{s_c{}^D} = r^P$.

Hence, at $(s_c{}^D, s_w{}^P, n^D)$, (r^P, k^P, t^P) is a solution with $s_c r = n$.

As for the stability of the solution, it is clear that the coefficients, γ_2, γ_1 and γ_o, of the characteristic equation are all positive. As in the proof of the previous proposition, for sufficiently small s_w, one has $\gamma_2 > n$. In addition, from (3.27), $\gamma_1 > \beta\delta_5$. Hence, $\gamma_2\gamma_1 > n\beta\delta_5 = \gamma_o$. Therefore, (r^P, k^P, t^P) is asymptotically stable. □

From this proposition, it is seen that if a D-path is stable, then there exists a stable P-path with a smaller s_w. In the light of the first 2 propositions, it is obvious that the converse of this proposition cannot be true. Remark that even if β is close to 0, there may exist an asymptotically stable P-path; but no stable D-path at all in such a situation.

4. Conclusions

Thus far, we built a dynamical model that reflects Keynes' effective demand principle *via* a simple Robinsonian animal spirits investment function, combined with Keynes-Wicksell adjustment process. We summarize the economic implication of our three propositions.

If the adjustment speed of prices in the product market is sufficiently fast compared to that of money wages in the labour market, irrespective of external parameters such as the rate of savings and the pattern of the production function, the path with $\frac{K_c}{K} = 0$ *is not asymptotically stable. Furthermore, a dynamically stable steady state growth path with full employment is, if ever it exists, only the path that guarantees* $s_c r = n$ (*Pasinetti's theorem*). *Conversely, if the adjustment speed in the product market is relatively slow, the dual theorem is true, depending on external parameters and the pattern of the production function.*

Bibliography

[1] Darity, W. A.(1981) "The Simple Analysis of Neo-Ricardian Growth and Distribution," *American Economic Reviews*, 71(5), pp.978-93.

[2] Kaldor, N. (1956) "Alternative Theories of Distributions," *Review of Economic Studies*, 23(2), pp.83-100.

[3] Meade, J. E. (1966) "The Outcome of Pasinetti's Process: A note," *Economic Journal*, 76(1), pp.161-5.

[4] Morishima, M. (1977), "Pasinetti's *Growth and Income Distribution* Revisited," *Journal of Economic Literature* 15(1), pp.56-61.

[5] Pasinetti, Luige L.(1962) "Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth," *Review of Economic Studies*, 29(4), pp.267-79.

[6] Pasinetti, Luige L.(1974) *Growth and Income Distribution*: *Essays in Economic Theory*, Cambridge University Press.

[7] Robinson, J. (1962) *Essays in the theory of Economic Growth*, Macmillan.

[8] Samuelson, P. A. and F. Modigliani(1966) "The Pasinetti Paradox in Neoclassical and More General Models," *Review of Economic Studies*, 33(4), pp.269-301.

[9] Stein, J. L. (1966) "Money and Capacity Growth," *Journal of Political Economy*, 74(5), pp.451-65.

[10] Stein, J. L. (1970) "Monetary Growth Theory in Perspective," *American Economic Review*, 60(1), pp.85-106.

[11] Torii, T. (1987) "A dynamic analysis of the macro distribution theory in connection with the Pasinetti paradox," *Economic Studies Quarterly* 38(3), pp.199-211. (in Japanese)